A guide for converting exponential growth to logistic growth

J.D. Yeakel

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1. Simple (Exponential) Growth

Assumptions

1. **Constant birth rate** (per capita): b
2. **Constant death rate** (per capita): d

Hence, per capita **net growth rate** r (i.e., "births minus deaths" per individual) is:

$$r = b - d.$$

Resulting Differential Equation

Let $N(t)$ be the population size at time t. Then exponential growth is modeled by:

$$\frac{dN}{dt} = r\,N.$$

- **Interpretation**: The larger the population, the faster it grows, with no upper bound.

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2. Adding More Realism: Density-Dependence

In nature, birth and death rates tend to change with population size:

1. **Per capita birth rate declines** as resources become limited, so we assume:

$$b' = b - aN \quad \text{(where } a \text{ is a positive constant)}$$

   - When N is small, $b' \approx b$.
   - As N gets larger, the term $-aN$ makes the **birth rate** go down.

2. **Per capita death rate increases** as crowding intensifies, so we assume:

$$d' = d + cN \quad \text{(where } c \text{ is a positive constant)}$$

   - When N is small, $d' \approx d$.
   - As N gets larger, the term $+cN$ makes the **death rate** go up.

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3. Net Per Capita Growth Rate

From these two modified rates:

$$\text{net per capita rate} = b' - d' = (b - aN) \; - \; (d + cN).$$

Simplify this:

$$b' - d' = (b - d) \; - \; (a + c)\,N.$$

Recalling that $r = b - d$, define $\alpha = a + c$. Then:

$$b' - d' = r - \alpha\,N.$$

Hence, **the net growth rate** for the whole population N is:

$$\frac{dN}{dt} = [r - \alpha N]\;N.$$

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4. The Logistic Growth Equation

Let's rearrange this into a more familiar form. Factor out r:

$$\frac{dN}{dt} = rN - \alpha N^2 = rN\big(1 - \tfrac{\alpha}{r}N\big).$$

Define the **carrying capacity** K by

$$K = \frac{r}{\alpha}.$$

Then

$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big).$$

This is the **logistic growth equation**.

Biological Interpretation

- r: intrinsic rate of increase (as in exponential growth).
- K: carrying capacity. Once N is near K, growth slows and eventually stops.

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5. Why This Matters

- **Initial Exponential Phase**: When N is small, $\frac{N}{K}$ is close to 0, so growth looks nearly exponential $(\frac{dN}{dt} \approx rN)$.
- **Resource Limitation**: As N grows, limited resources and increased crowding reduce births, increase deaths, and slow the growth.
- **Equilibrium**: In the long run, the population stabilizes around $N = K$, where births and deaths balance out.

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6. Recap

1. **Start with** a simple exponential model, $\frac{dN}{dt} = rN$.
2. **Acknowledge limitations**: constant birth and death rates ignore environmental constraints.
3. **Incorporate density-dependence**:
   - Birth rates **decrease** with N.
   - Death rates **increase** with N.
4. **Combine** these rates to find a net per capita rate of $r - \alpha N$.
5. **Obtain** $\frac{dN}{dt} = rN - \alpha N^2$.
6. **Rewrite** in logistic form: $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$.

That's it! Understanding each step helps clarify why real populations don't grow forever but instead level off at a carrying capacity.