

Susceptible individuals
 Infected individuals
 Recovered individuals

S
 I
 R

become infected via contact
 w/ infected individuals

Recover at a ~~fixed~~ fixed rate γ

Total number of individuals
 does not change over time

$$S + I + R = N$$

So: $\frac{1}{\gamma}$ is the average amount
 of time an individual is
 infected

$$\frac{dS}{dt}, \frac{dI}{dt}, \frac{dR}{dt}$$

$$\frac{dN}{dt} = 0 \Rightarrow \frac{d(S+I+R)}{dt} = 0$$

$$\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0$$

λ = the force of infection ~ the per-capita rate at which susceptible
 individuals acquire infection. ~~(*)~~ λ is not constant
 λ changes with I

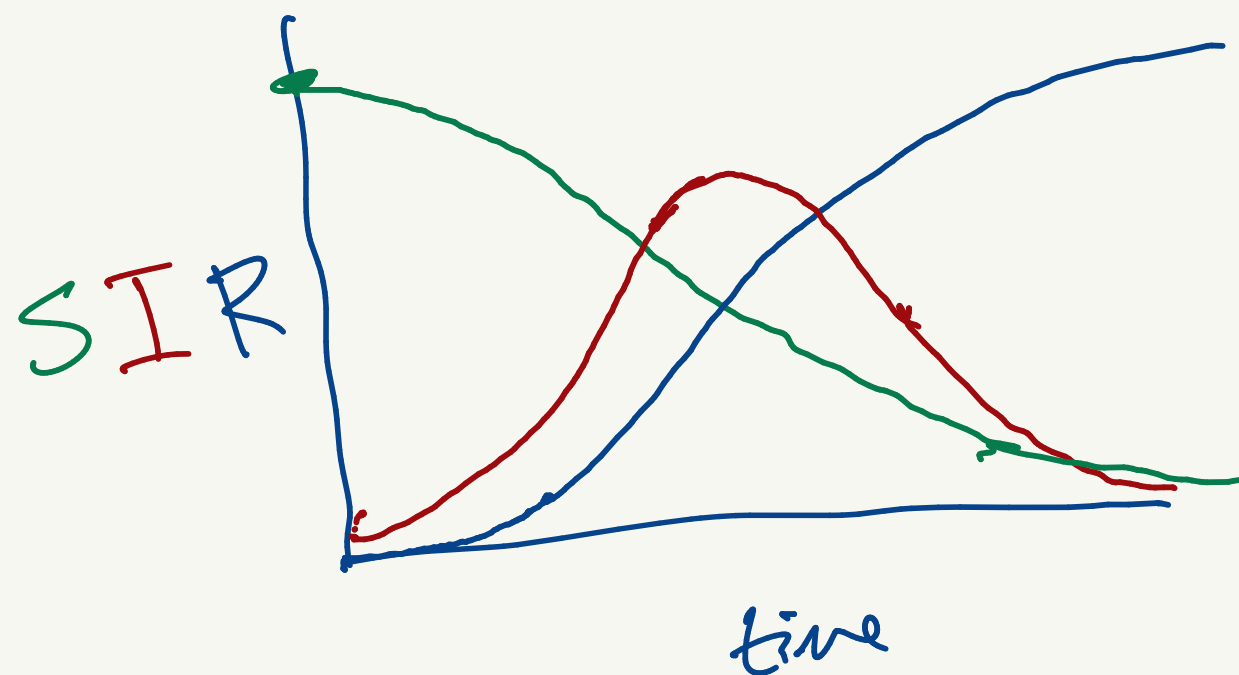
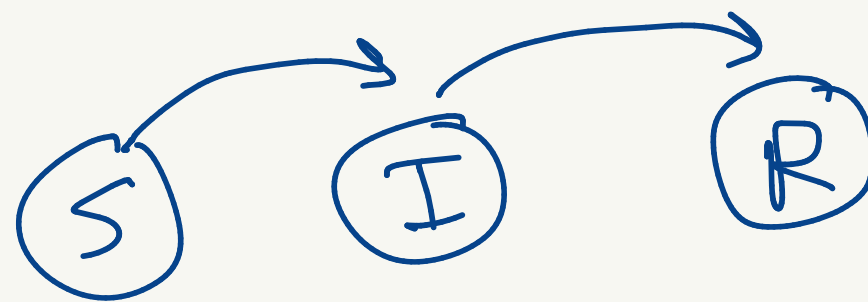
$$\lambda(I) = \underbrace{\beta \frac{I}{N}}_{\text{transmission rate } \frac{1}{[\text{time}]}} \underbrace{\}_{\text{interaction term}}$$

$$\frac{dS}{dt} = - \underbrace{\lambda(I)S} = - \beta \frac{I}{N} S$$

$$\lambda(I) = \beta \frac{I}{N}$$

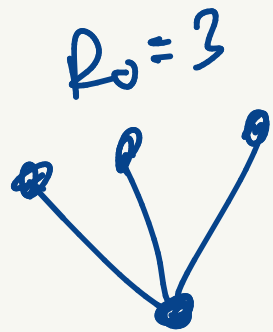
$$\frac{dI}{dt} = \underbrace{\lambda(I)S} - \underbrace{\gamma I} = \beta \frac{I}{N} S - \gamma I$$

$$\frac{dR}{dt} = \underbrace{\gamma I}$$



The basic reproductive number R_0

R_0 = the average number of secondary cases arising from a typical primary case in an entirely susceptible population



$$R_0 > 1$$

pathogen expected to spread

$$R_0 < 1$$

pathogen is expected to decline

We assume that the ^{host} population @ $t=0$ has 1 infected individual

⊗ The disease will fail to spread if $\frac{dI}{dt} < 0$ at $t=0$

$S(0) = N-1$
 $I(0) = 1$

$$\beta \frac{I}{N} S - \gamma I < 0$$

$$\beta \frac{I}{N} S < \gamma I$$

transmission rate

$$\rightarrow \frac{\beta}{\gamma} \frac{S}{N} < 1$$

recovery rate

$$\frac{\beta}{\gamma} \frac{N-1}{N} < 1$$

$$\frac{\beta}{\gamma} < 1$$

$$R_0 = \frac{\beta}{\gamma}$$

if $\frac{\beta}{\gamma} < 1$ no spread

if $\frac{\beta}{\gamma} > 1$ outbreak

$$R_0 = \begin{bmatrix} \beta \\ \gamma \end{bmatrix} \begin{cases} > 1 & \text{outbreak} \\ < 1 & \text{pathogen dies out} \end{cases}$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I - \mu I$$

R_0

Flu: 2-3

HIV/AIDS: 3-5

Smallpox: 5-7

Ebola: 1.5-2.5

Measles: 12-18

$$0 = \beta \frac{I}{N} S - \gamma I - \mu I$$

$$\beta \frac{I}{N} S = \gamma I + \mu I$$

$$\beta \frac{S}{N} = \gamma + \mu \rightarrow \beta \frac{N-1}{N} = \gamma + \mu$$

$$R_0 = \frac{\beta}{\gamma + \mu} = 1$$

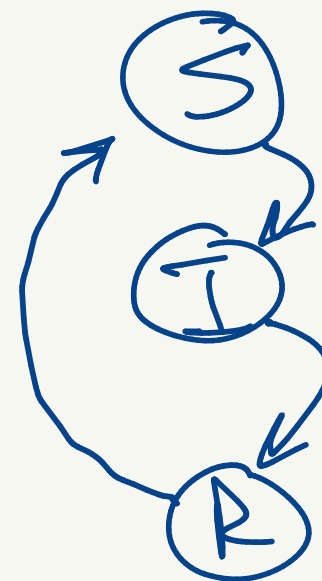
R_0 if we consider

Recovery
Mortality

$$\frac{N-1}{N} \approx 1 \text{ if } N \text{ is large}$$

$$\frac{99999}{100000} = 0.99999 \approx 1$$

Now let's consider the SIR model w/
demographic processes



Assume

1) All individuals can reproduce at rate b

2) All offspring are susceptible

3) All individuals experience mortality at rate ν
independent of disease!

$$\frac{dS}{dt} = \overbrace{bN} - \beta \frac{I}{N} S - \nu S = b(S+I+R) - \beta \frac{I}{N} S - \nu S$$

$$\frac{dI}{dt} = \beta \frac{I}{N} S - \gamma I - \nu I$$

$$\frac{dR}{dt} = \gamma I - \nu R$$

$R_0?$

$$\boxed{\frac{dI}{dt} > 0}$$

$$\beta \frac{I}{N} S > \gamma I + \nu I$$

$$\beta \frac{S}{N} > \gamma + \nu$$

$$\boxed{\frac{\beta}{\gamma + \nu} > 1}$$

$$R_0 = \frac{\beta}{\gamma + \nu}$$

↑ ↑

- Small modification (accounts for an altered rate out of the infected state: Recovered + Dead) to R_0

- Not the full story!

Solve for the steady state!

$$\left[\begin{array}{l} \frac{dS}{dt} = 0, \quad \frac{dI}{dt} = 0, \quad \frac{dR}{dt} = 0 \\ 0 = \beta \frac{I}{N} S - \gamma I - \nu I \\ 0 = \gamma I - \nu R \end{array} \right]$$

$$\frac{dS}{dt} = 0 = bN - \beta \frac{I}{N} S - \nu S$$

$$bN = \beta \frac{I}{N} S + \nu S$$

$$bN = \frac{S}{N} (\beta I + N\nu)$$

$$bN = \left(\frac{\gamma + \nu}{\beta} \right) (\beta I + N\nu)$$

$$\left(\frac{1}{N} \right) \frac{bN\beta}{\gamma + \nu} = (\beta I + N\nu) \left(\frac{1}{N} \right)$$

$$\frac{b\beta}{\gamma + \nu} = \beta \frac{I}{N} + \nu \rightarrow \beta \frac{I}{N} = \frac{b\beta}{(\gamma + \nu)} - \nu$$

$$0 = \beta \frac{I}{N} S - \gamma I - \nu I$$

$$\beta \frac{I}{N} S = \gamma I + \nu I$$

$$\beta \frac{S}{N} = \gamma + \nu$$

$$\left\{ \frac{S}{N} = \frac{\gamma + \nu}{\beta} \right\}$$

proportion of susceptibles

$$\left\{ \frac{I}{N} = \frac{b}{(\gamma + \nu)} - \frac{\nu}{\beta} \right\}$$

pop. of infected

$$\gamma I = \nu R$$

$$R = \frac{\gamma}{\nu} I$$

$$\frac{R}{N} = \frac{\gamma}{\nu} \frac{I}{N}$$

$$\left\{ \frac{R}{N} = \frac{\gamma}{\nu} \left[\frac{b}{(\gamma + \nu)} - \frac{\nu}{\beta} \right] \right\}$$

Pop. of recovered

Not zero... not disease-free.

- 1) This means the disease becomes ENDEMIC ($I^* > 0$)
- 2) Proportions of $\frac{S}{N}$, $\frac{I}{N}$, $\frac{R}{N}$ at steady state depends on both demographic and disease parameters

What does R_0 mean?

R_0 = the average number of people each sick person infects

$R_0 = 1$: $\textcircled{1} \rightarrow \textcircled{1}$

$R_0 = 2$: $\textcircled{0} \begin{matrix} \nearrow \textcircled{0} \\ \searrow \textcircled{0} \end{matrix}$

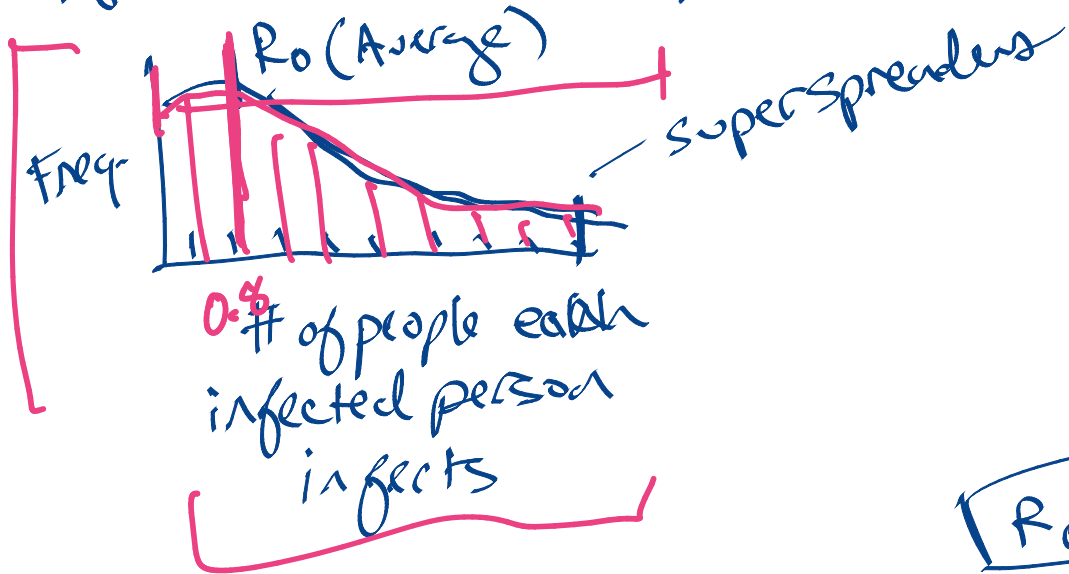
$R_0 = < 1$: $\textcircled{0} \rightarrow \textcircled{0}$
 $0.5 \nearrow \textcircled{0}$

$\left\{ \begin{array}{l} R_0 > 1 \text{ epidemic continues} \\ \text{exponential growth} \\ R_0 < 1 \text{ epidemic decline} \end{array} \right.$

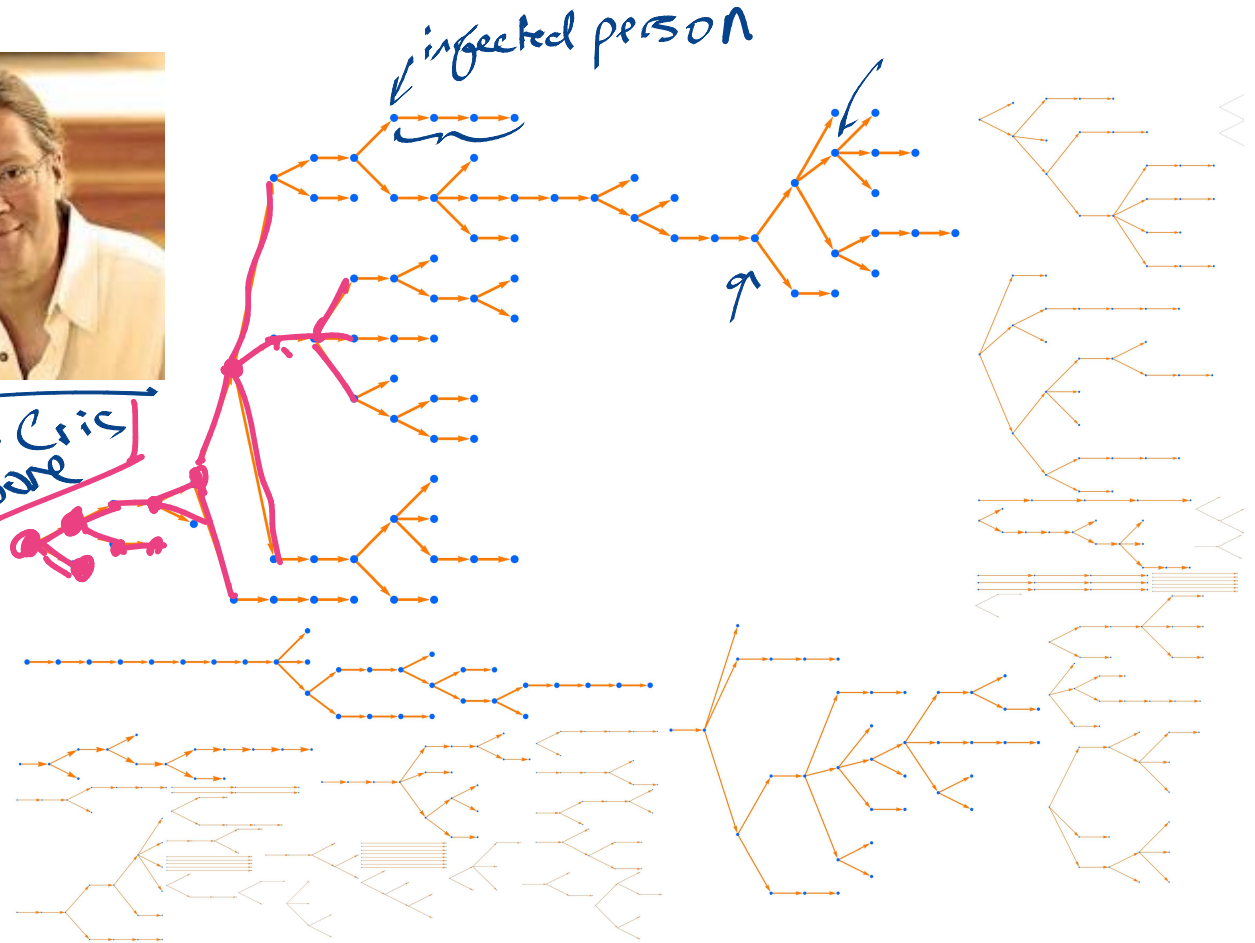
What does R_0 mean?

- virus life cycle
- human physiology
- human social dynamics

R_0 ~ describes the AVERAGE



~~Chris~~ Chris Moore



> A hundred random outbreaks in a scenario where each sick person interacts with 10 others, and infects each one with probability 8 percent. Here $R_0 = 0.8$ and the average outbreak size is five, but 1 percent of the outbreaks have size 50 or larger, and in this run the largest has size 82.

$$R_0 < 1$$

superspreaders

What does R_0 mean?



$$R_0 = 0.8$$

A hundred random outbreaks in a scenario with superspreading, where 1 percent of the cases infect 20 others. As in Figure 1, we have $R_0 = 0.8$ and the average outbreak size is 5, but now the heavy tail of outbreaks is much heavier. In this run the largest outbreak has size 663.

