Interspecies interactions → Predation

Predator < specialist, generalist

Prey

Lotka Volterra Model

$$\frac{dN}{dt} = rN - aNP$$

$$\frac{dP}{dt} = baNP - mP$$

r ~ prey growth rate

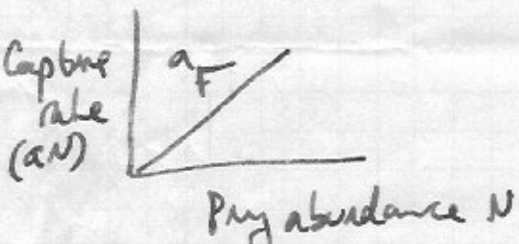
a ~ capture efficiency of prey

aN = # of prey killed per unit time (capture rate)

b ~ efficiency that prey mass is turned over to predator mass

m ~ Natural mortality of predators

(*) Linear functional response



→ The more prey, the greater rate of capture

Find F.P.!

$$rN = aNP \rightarrow r = aP \rightarrow P^* = \frac{r}{a}$$ } when does $P^* \uparrow, \downarrow$

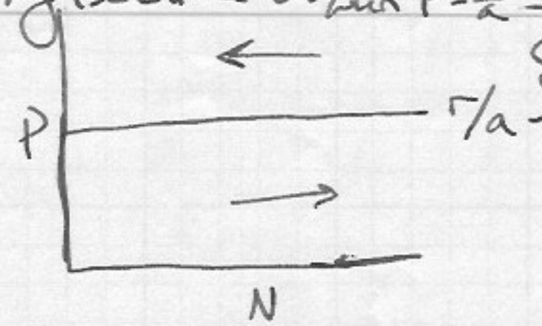
$$baNP = mP$$

$$\hookrightarrow baN\left(\frac{r}{a}\right) = m\left(\frac{r}{a}\right)$$

$$N^* = \frac{m}{ba}$$ } when does $N^* \uparrow, \downarrow$?

Now we have 2 isoclines: one for prey, one for preds

8.2 Prey Isocline $\frac{dN}{dt} = 0$ when $P = \frac{r}{a}$



Decline

$\frac{dN}{dt} < 0$ when $rN - aNP < 0$

$rN < aNP$

$r < aP$

$\frac{r}{a} < P$ (or) $P > \frac{r}{a}$

Growth

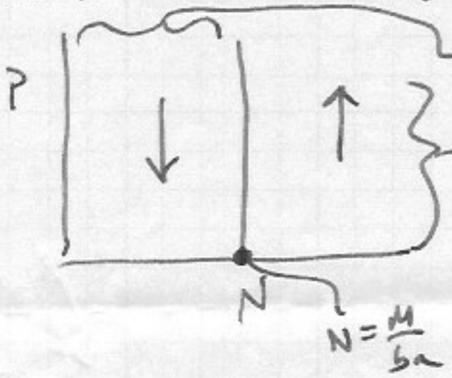
$\frac{dN}{dt} > 0$ when $rN - aNP > 0$

$rN > aNP$

$r > aP$

$r/a > P$ (or) $P < \frac{r}{a}$

Predator Isocline $\frac{dP}{dt} = 0$ when $N = \frac{m}{ba}$



Decline

$\frac{dP}{dt} < 0$ when $baNP - mP < 0$

$baNP < mP$

$baN < m$

$N < \frac{m}{ba}$

Growth

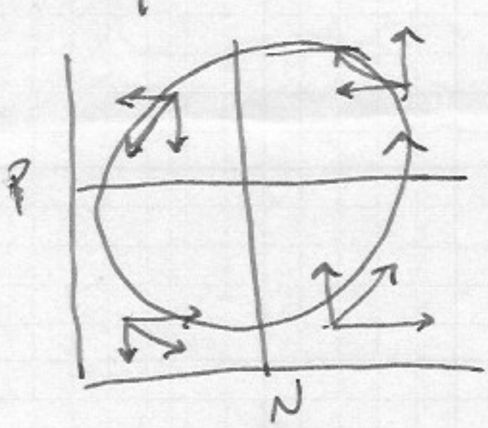
$\frac{dP}{dt} > 0$ when $baNP - mP > 0$

$baNP > mP$

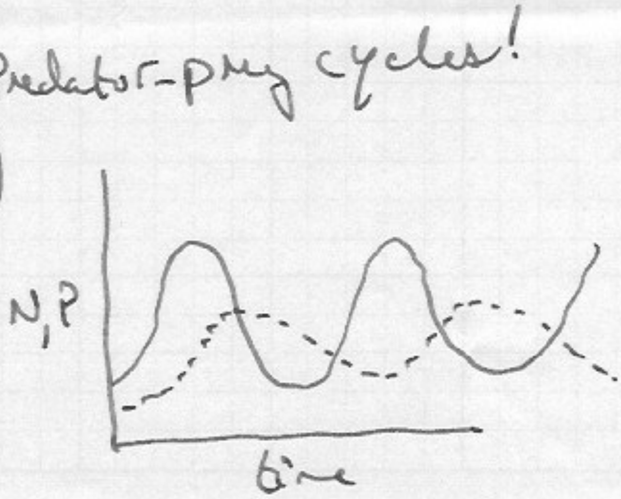
$baN > m$

$N > \frac{m}{ba}$

Combine Isoclines to uncover dynamics



Predator-prey cycles!

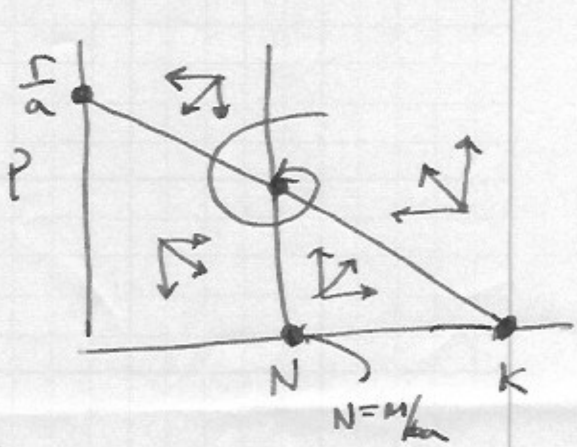


★ Different initial condition leads to ~~different~~ cycles w/ different amplitudes

Do the same thing for

$$\frac{dN}{dt} = rN\left(1-\frac{N}{k}\right) - aNP$$

m $$\frac{dP}{dt} = baNP - mP$$



Density-dependent regulation of prey suppresses the cycling.

— How can we learn about the NATURE of a fixed point? Attractor, Repellor, Cycles?

Given $\dot{x} = f(x,y)$
$\dot{y} = g(x,y)$ we first solve (x^*, y^*)

Next we study the dynamics around (x^*, y^*)

Specifically, do perturbations grow or decay

Define perturbations:

$$\epsilon = x - x^*$$
$$\eta = y - y^*$$

$$\dot{\epsilon} = \dot{x} = f(x,y) = \overbrace{f(x^*+\epsilon, y^*+\eta)}^{\text{Taylor Expand}}$$

$$\dot{\epsilon} = f(x^*, y^*) + \epsilon \frac{\partial f}{\partial x}\bigg|_{(x^*,y^*)} + \eta \frac{\partial f}{\partial y}\bigg|_{(x^*,y^*)} + \text{h.o.t.}$$

$$\dot{\epsilon} = \epsilon \frac{\partial f}{\partial x}\bigg|_* + \eta \frac{\partial f}{\partial y}\bigg|_* + \text{h.o.t.}$$

8.4

Start

We have:

$$\dot{\epsilon} = \epsilon \frac{\partial f}{\partial x}\Big|_* + \eta \frac{\partial f}{\partial y}\Big|_* + \text{h.o.t.}$$

Similarly,

$$\dot{\eta} = g(x^* + \epsilon, y^* + \eta) = g(x^*, y^*) + \epsilon \frac{\partial g}{\partial x}\Big|_* + \eta \frac{\partial g}{\partial y}\Big|_* + \text{h.o.t.}$$

$$\dot{\eta} = \epsilon \frac{\partial g}{\partial x}\Big|_* + \eta \frac{\partial g}{\partial y}\Big|_* + \text{h.o.t.}$$

We've linearized the problem around the fixed point.

In matrix notation:

$$\begin{pmatrix} \dot{\epsilon} \\ \dot{\eta} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x}\Big|_* & \frac{\partial f}{\partial y}\Big|_* \\ \frac{\partial g}{\partial x}\Big|_* & \frac{\partial g}{\partial y}\Big|_* \end{pmatrix}}_{\text{Jacobian Matrix } J} \begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$$

Ex) given $\dot{x} = -x + x^3$, $\dot{y} = -2y$

a) find all F.P.

b) Linearize around fixed points given $\begin{pmatrix} \dot{\epsilon} \\ \dot{\eta} \end{pmatrix} = J \begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$

$$f(x) = -x + x^3 = \emptyset \rightarrow x(x^2 - 1) = \emptyset$$

$$g(y) = -2y = \emptyset \qquad \hookrightarrow x(x-1)(x+1) = \emptyset$$

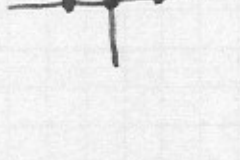
$$\begin{cases} x^* = 1, -1, \emptyset \\ y^* = \emptyset \end{cases}$$

F.P.

$(1, \emptyset)$

$(-1, \emptyset)$

$(\emptyset, \emptyset)$



8.5

(b) Linearize the system:

$$J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} -1+3x^2 & \varnothing \\ \varnothing & -2 \end{pmatrix}$$

Now - examine near F.P.

Near $(1,\varnothing)$ $\qquad \begin{pmatrix} \dot\epsilon \\ \dot\eta \end{pmatrix} = \begin{pmatrix} 2 & \varnothing \\ \varnothing & -2 \end{pmatrix}\begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$

Near $(\varnothing,\varnothing)$ $\qquad \begin{pmatrix} \dot\epsilon \\ \dot\eta \end{pmatrix} = \begin{pmatrix} -1 & \varnothing \\ \varnothing & -2 \end{pmatrix}\begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$

Near $(-1,\varnothing)$ $\qquad \begin{pmatrix} \dot\epsilon \\ \dot\eta \end{pmatrix} = \begin{pmatrix} 2 & \varnothing \\ \varnothing & -2 \end{pmatrix}\begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$

(c) Solutions to Linear Systems

In 1D we had $\dot\eta = \frac{\partial f}{\partial x}\eta$, and solution was

$\eta(t) = \eta_0 e^{\lambda t}$ where $\lambda = \frac{\partial f}{\partial x}$

In 2-D (really, n-D), we have the general linear form

$\dot{\vec\eta} = J\eta$ and we seek solutions of the form:

$\vec\eta = \vec v e^{\lambda t}$ $\qquad \vec v$ = eigenvector

$\boxed{\lambda = \text{eigenvalue}}$ — we will focus on this

$\dot{\vec\eta} = \vec v \lambda e^{\lambda t}$

(see 8.6)

so: [illegible]

$\dot{\vec\eta} = \lambda\vec\eta$ $\quad$ b/c $\quad \vec\eta = \vec v e^{\lambda t}$

[illegible]

and: $\dot{\vec\eta} = \lambda\vec\eta = J\vec\eta = J\vec v e^{\lambda t}$

so: J

We seek solutions in form
$\eta(t) = e^{\lambda t}\vec{v} \rightarrow \dot{\vec{\eta}} = \lambda e^{\lambda t}\vec{v}$

We currently have $\dot{\vec{\eta}} = J\eta \rightarrow J\vec{v}e^{\lambda t}$

so: $\lambda e^{\lambda t}\vec{v} = e^{\lambda t}J\vec{v}$

$J\vec{v} = \lambda\vec{v}$ } if $\vec{v}$ is an eigenvector of J, with corresponding eigenvalue λ, then linear solutions exist.

Focus on eigenvalues...

KEY [The eigenvalues of $J|_*$ will give us the growth rates of the linear approximation to dynamics around the fixed point (*).

How do you find the eigenvalues of J?
↳ given by the characteristic equation: $\det(J-\lambda I)=$

What is $J-\lambda I$? if $J = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $J-\lambda I = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$

What is $\det(\cdot)$? for matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, det. is $ad-bc$

so: $\det(J-\lambda I) = (a-\lambda)(d-\lambda) - bc$

Another characteristic of matrices is Trace: $a+d$ for 2×2 matrix

$\tau = \text{trace}(\mathcal{J}) = a + d$

$\Delta = \det(\mathcal{J}) = ad - bc$

then $\lambda_1 = \dfrac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}$, $\lambda_2 = \dfrac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$

- Notice there are 2 eigenvalues for the 2-D system, and they depend only on the trace and determinant of $\mathcal{J}$!

- Eigenvalues are distinct (generally), and eigenvectors are linearly independent, which allows us to write general solution for $\eta(t)$ as

$$\vec{\eta}(t) = c_1 e^{\lambda_1 t} \vec{v_1} + c_2 e^{\lambda_2 t} \vec{v_2}$$ where c_1, c_2 are constants.

Consider

L-V pred prey system:

$$\frac{dN}{dt} = rN - aNP = f(N, P)$$

$$\frac{dP}{dt} = baNP - mP = g(N, P)$$

There are 2 fixed points: $(\emptyset, \emptyset)$

$\left(\frac{m}{ba}, \frac{r}{a}\right)$

What is the Jacobian?

$$\mathcal{J} = \begin{pmatrix} \frac{\partial f}{\partial N} & \frac{\partial f}{\partial P} \\ \frac{\partial g}{\partial N} & \frac{\partial g}{\partial P} \end{pmatrix}\Bigg|_* = \begin{pmatrix} r - aP & -aN \\ baP & baN - m \end{pmatrix}$$

8.8

$$J = \begin{pmatrix} r - aP & -aN \\ baP & baN - m \end{pmatrix}$$

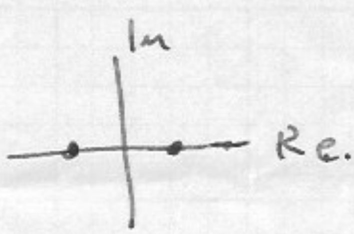
$$J\Big|_{(0,0)} = \begin{pmatrix} r & 0 \\ 0 & -m \end{pmatrix} \qquad J\Big|_{\left(\frac{m}{ba}, \frac{r}{a}\right)} = \begin{pmatrix} r - a\left(\frac{r}{a}\right) & -a\left(\frac{m}{ba}\right) \\ ba\left(\frac{r}{a}\right) & ba\left(\frac{m}{ba}\right) - m \end{pmatrix}$$

$$J\Big|_{\left(\frac{m}{ba}, \frac{r}{a}\right)} = \begin{pmatrix} 0 & -\frac{m}{b} \\ br & 0 \end{pmatrix}$$

Find Eigenvalues

$$\lambda_{1,2}\Big|_{(0,0)} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2} = \frac{1}{2}\left(r - m \pm \sqrt{(m+r)^2}\right)$$

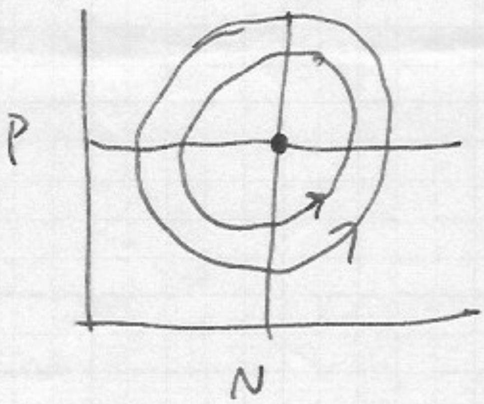
High ~~Positive~~ branch: $\lambda_1 > 0$
Low branch: $\lambda_2 < 0$



$$\lambda_{1,2}\Big|_{\left(\frac{m}{ba}, \frac{r}{a}\right)} = \pm\sqrt{-mr} = \pm i\sqrt{mr}$$ } eigenvalues purely imaginary

$$\begin{cases} \Delta = mr \\ \tau = 0 \end{cases}$$

From graphical analysis:



(N^*, P^*) is a center

"Centers"

- any initial condition leads to a unique cycle

8.9

In 2-D, this is always true:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

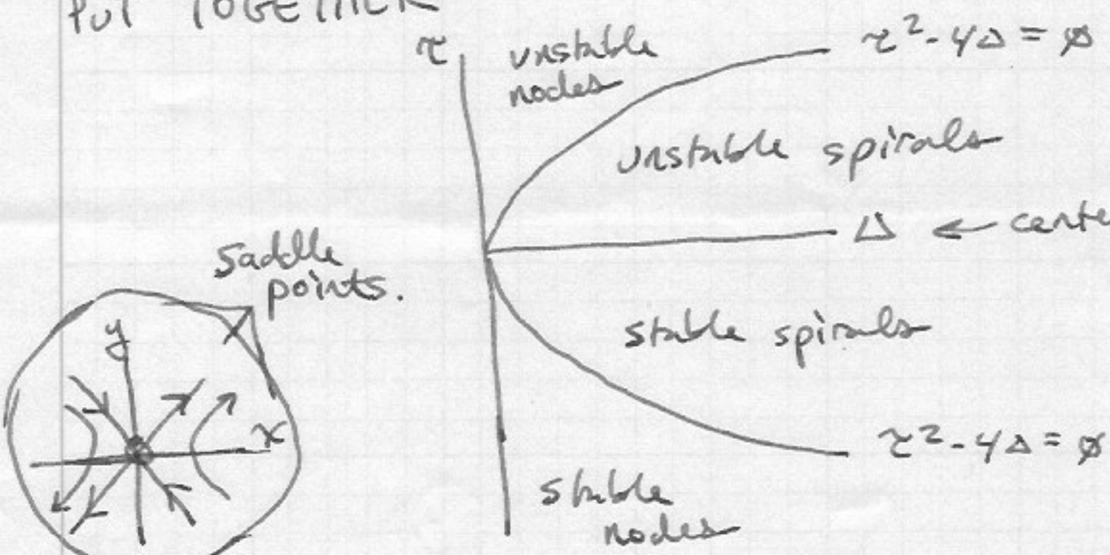
$\Delta = \lambda_1 \lambda_2$

$\tau = \lambda_1 + \lambda_2$

- if one eigenvalue is (+), and one is negative, Δ is (−)
- if both eigs are (+, +) or (−, −), Δ is ~~p~~ (+)
  - ↳ ~~necess~~ Stability requires that Δ is (+), but is not a sufficient condition to ~~guu~~ guarantee it.

- When $\tau < \emptyset$, both eigenvalues have negative real parts, so ~~th~~ fixed point is STABLE if $\Delta > \emptyset$
- When $\tau > \emptyset$, and $\Delta > \emptyset$, both eigenvalues ~~are~~ have positive real parts ~ UNSTABLE
- if $\tau^2 - 4\Delta < \emptyset$, there are imaginary parts to eigenvalues
  - ↳ cycles

PUT TOGETHER



(illustration of saddle point)