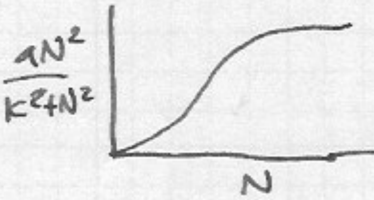
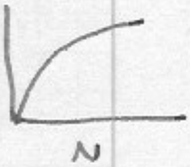
Generalized Modeling of biological System

Density dependent growth of a single species

$$\frac{d}{dt}N = \underbrace{\frac{aN^2}{k^2+N^2}} - bN \qquad \text{OR} \qquad \frac{dN}{dt} = \underbrace{\frac{aN}{1+N}}_{\text{Type II}} - bN$$



- Type III
- growth is very low when N is very low
- Maximized (saturated) at N increases.
- Reproductive success requires 'aggregates'



Generalize the problem

$$\frac{d}{dt}N = S(N) - D(N)$$

★ This may be more accurate in terms of our knowledge of the system...

But the problem is in the analysis!

1) Solve for F.P.

$$\emptyset = S(N) - D(N)$$

$S(N) = D(N)$ — can't solve for N^*!

Define a new variable N^*

1) It is assumed to be positive
2) Not a placeholder, for a value to be filled in later, but a formula surrogate for
   EVERY POSITIVE STEADY state in the class of systems represented by $\dot{N} = S(N) - D(N)$

What determines the stability of N^*?

The derivative... which we will now start calling an eigenvalue:

$$\lambda = \frac{\partial}{\partial N}\dot{N} = \underbrace{\left.\frac{\partial S(N)}{\partial N}\right|_*} - \underbrace{\left.\frac{\partial D(N)}{\partial N}\right|_*}$$

These are unknown parameters in the system.

BUT hard to interpret biologically (slope of function @ fixed point)

we will use the identity:

$$\left.\frac{\partial S(N)}{\partial N}\right|_* = \frac{S(N^*)}{N^*} \left.\frac{\partial \log S(N)}{\partial \log N}\right|_* \Bigg\} \text{ holds for all } \begin{cases} S(N^*) > 0 \\ N^* > 0 \end{cases}$$

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$$\left.\frac{\partial S(N)}{\partial N}\right|_* = \frac{S(N^*)}{N^*} \left.\frac{\partial \log S(N)}{\partial \log N}\right|_* \quad \text{multiply by } \frac{\partial S(N)}{\partial S(N)}$$

$$= \frac{S(N^*)}{N^*} \frac{\partial \log S(N)}{\partial S(N)} \frac{\partial S(N)}{\partial \log N}$$

$$* \text{ where } \left.\frac{\partial \log S(N)}{\partial S(N)}\right|_* = \frac{1}{S(N^*)}$$

(Proof of the relationship)

$$= \frac{1}{N^*} \left.\frac{\partial S(N)}{\partial \log N}\right|_* \quad \text{multiply by } \frac{\partial N}{\partial N}$$

$$= \underbrace{\frac{1}{N^*} \frac{\partial N}{\partial \log N}}\left.\frac{\partial S(N)}{\partial N}\right|_*$$

Define $N = e^u$ or $\log N = u$

$$\left.\frac{\partial N}{\partial (\log N)}\right|_* = \left.\frac{\partial e^u}{\partial \log e^u}\right|_* = \left.\frac{\partial e^u}{\partial u}\right|_* = e^u\Big|_* = N^*$$

$$= \frac{1}{N^*} N^* \left.\frac{\partial S(N)}{\partial N}\right|_* = \left.\frac{\partial S(N)}{\partial N}\right|_* \quad \text{QED}$$

<u>Back to the problem</u>

$$\left.\frac{\partial S(N)}{\partial N}\right|_* = \frac{S(N^*)}{N^*} \left.\frac{\partial \log S(N)}{\partial \log N}\right|_* \quad \underline{\text{and}} \quad \left.\frac{\partial D(N)}{\partial N}\right|_* = \frac{D(N^*)}{N^*} \cdot \frac{\partial \log D}{\partial \log}$$

Substitute to get! [also, notation is: $S(N^*) = S^*$]

$$\lambda = \frac{S^*}{N^*} \frac{\partial \log S}{\partial \log N} - \frac{D^*}{N^*} \frac{\partial \log D}{\partial \log N}$$

These are constants b/c functions evaluated @ steady state N^*