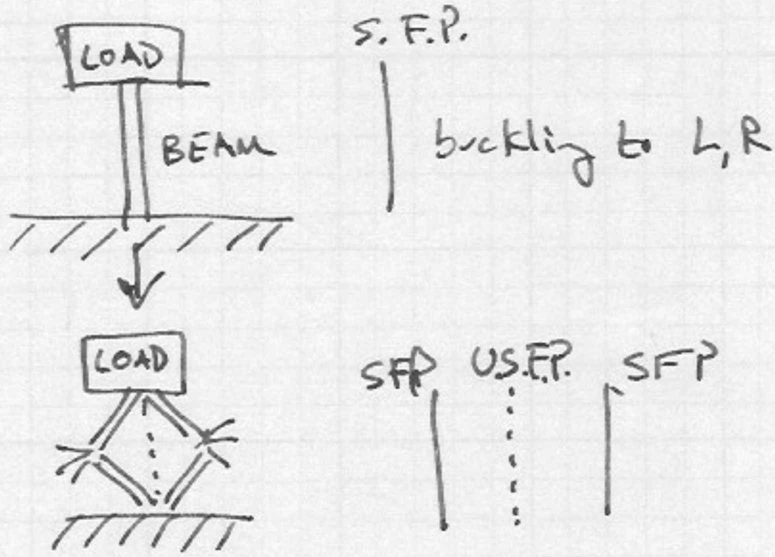
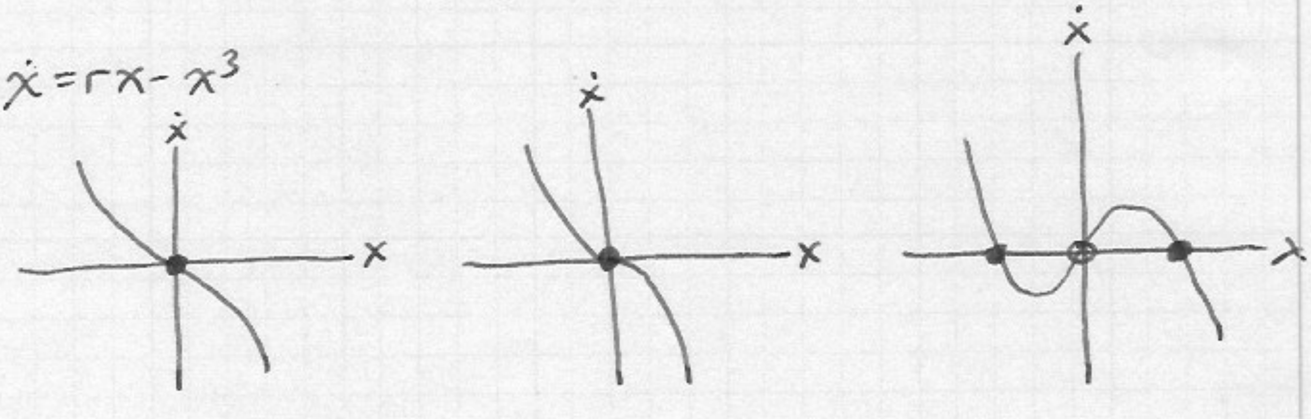
The pitchfork bifurcation

- Common in systems with symmetry

Spatial (L, R) symmetry ~ F.P. (dis)appear in pairs.



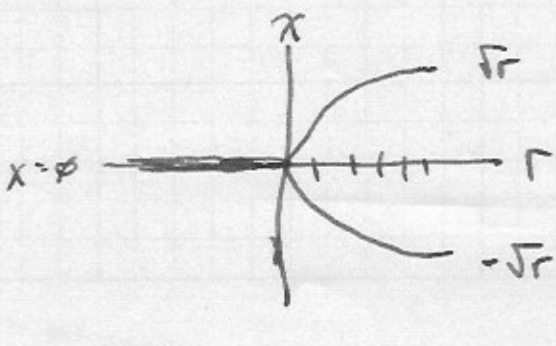
$\dot{x} = rx - x^3$



$r < \phi$

F.P.: $\quad \phi = rx - x^3 = x(r - x^2)$

~~xxx~~ $\quad$ F.P. $\quad x = \phi, \; x = \pm\sqrt{r}$



<u>Stability</u>

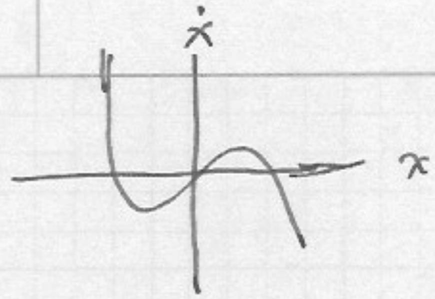
$$\frac{\partial f(x)}{\partial x} = r - 3x^2$$

$$r - 3x^2 \Big|_{x=\phi} \to r$$

$$\Big|_{x=\frac{-\sqrt{r}}{\sqrt{r}}} \to r - 3r = -2r$$

6.1b

$\dot{x} = rx - x^3$

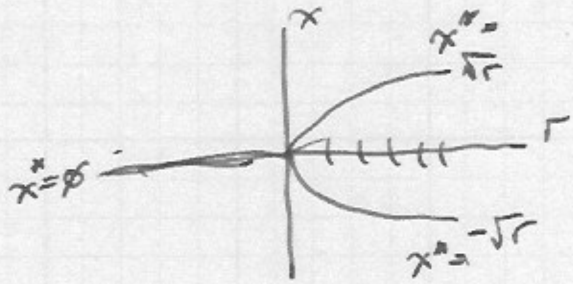


F.P.: $\phi = rx - x^3$

$x(r - x^2)$

$x^* = \phi$

$x^* = \pm\sqrt{r}$



Stability

$$\frac{\partial f(x)}{\partial x} = r - 3x^2$$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{x^* = \phi} = r$$

~~positive at r~~
Stable when ~~...~~ $r < \phi$
Unstable when $r > \phi$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{(x^* = \sqrt{r})} = r - 3(\sqrt{r})^2 = r - 3r = -2r$$

$$\left.\frac{\partial f(x)}{\partial x}\right|_{(x^* = -\sqrt{r})} = r - 3(-\sqrt{r})^2 = r - 3r = -2r$$

Stable when $r > \phi$
Unstable when $r < \phi$

Bifurcation occurs @

$x^* = \cancel{\phi}\ \phi$ or $\pm\sqrt{r}$

and $\frac{\partial f(x)}{\partial x} = \phi$

$\Leftrightarrow \phi = r - 3x^2 \rightarrow r_c = 3x^2$

~~$3x^2 = r$~~

~~$x^2 = \frac{r}{3}$~~

~~$x = \pm\sqrt{\frac{r}{3}}$~~

given $x_c^* = \phi$ ...

$r_c = \phi$

given $x_c^* = \pm\sqrt{r}$

$r_c = 3(\pm\sqrt{r})^2$

$r_c = 3r_c$

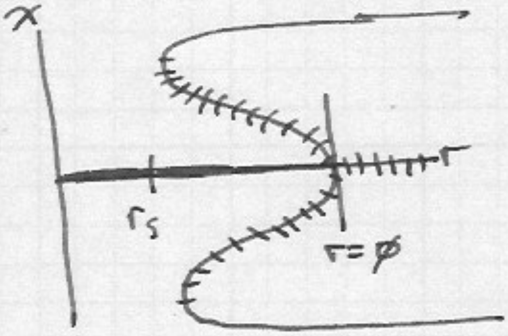
$r_c - 3r_c = \phi$

$r_c(1-3) = \phi \rightarrow r_c = \phi$

6.2

Consider the system
$$\dot{x} = rx + x^3 - x^5$$
Solve for the fixed points as a function of r: $x^*(r)$

to get



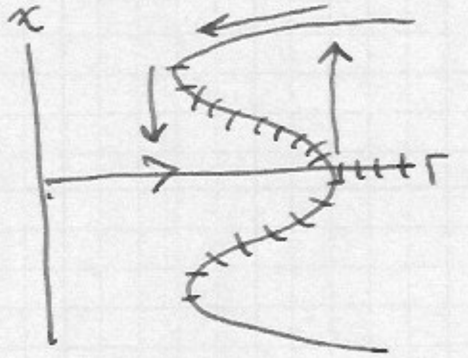
- $x^* = \phi$ is locally stable for $r < \phi$
- x^5 term requires that the unstable branches turn around and become stable at $r = r_s$
- In the range $r_s < r < \phi$, 2 qualitatively different ~~fixed points~~ stable states exist (origin, large amplitude fixed points)

  ⊛ $\boxed{\text{ALTERNATIVE STABLE STATES}}$

  ↳ The origin is <u>locally stable</u> to small perturbations, but not globally stable (consider perturbations of different sizes for different values of 'r').

- Existence of different alternative stable states allows for the possibility of jumps or hysteresis as r is varied.
  - Start @ $x^* = \phi$, increase r
  - ~~Fixed~~ Stable state $\boxed{\text{jumps}}$ @ $r = \phi$
  - Decrease r, but stable state does not immediately jump back to the origin

6.3



- r has to be lowered much further than expected to recover original stable state.

=> hysteresis

- Note that bifurcation @ fixed point is a saddle node

Spruce budworm example! ~ attacks leaves of balsam fir trees

goal: model interaction btw. budworms & the forest

↳ budworms evolve on fast timescales ~ months
trees grow & die on slow timescales ~100 years

↳ Sol.: treat forest variables as constants.
↳ or to drift slowly

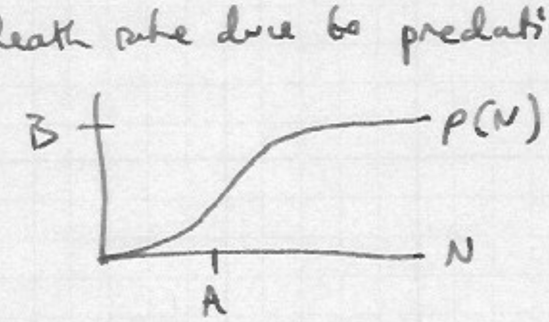
Budworm dynamics

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - p(N)$$

K depends on amt of foliage left (drifts slowly)

p(N) death rate due to predation (birds)

↳



- hard to find when rare
- saturate birds appetites @ p(N) = B
- A is the density @ which this shifts

$$p(N) = \frac{BN^2}{A^2+N^2}$$

Type III functional form

6.4

$$\dot{N} = RN\left(1-\frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}$$

We can simplify the model by casting it in dimensionless form:

Dimensions

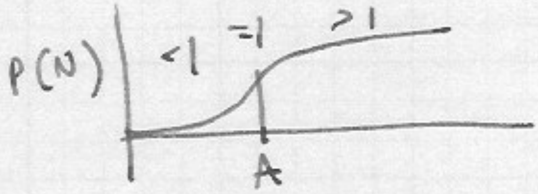
$$\begin{cases} N = \text{\# of budworms} \\ K \sim \\ A \sim \end{cases}$$

$x = \frac{N}{A}$ so: $xA = N$

$$\frac{d}{dt} xA = RxA\left(1-\frac{xA}{K}\right) - \frac{Bx^2A^2}{A^2+x^2A^2}$$

so a dimensionless term ~~that~~ is $\frac{N}{A}$

$$A\frac{dx}{dt} = RxA\left(1-\frac{xA}{K}\right) - \frac{Bx^2}{1+x^2}$$



$$\frac{A}{B}\frac{dx}{dt} = \frac{R}{B}Ax\left(1-\frac{xA}{K}\right) - \frac{x^2}{1+x^2}$$

$N/A \sim$ if $= 1$, N is @ A
if < 1, N is below A
if > 1, N is above A

Dimensionless time variable

$$\tau = \frac{Bt}{A}$$

(same w/ ~~=~~ N/K)

$$r = \frac{RA}{B} = \frac{[\frac{1}{t}][N]}{[\frac{N}{t}]} \sim \text{Dimensionless}$$

$$\kappa = \frac{K}{A}$$

so: $$\frac{dx}{d\tau} = rx\left(1-\frac{x}{\kappa}\right) - \frac{x^2}{1+x^2}$$

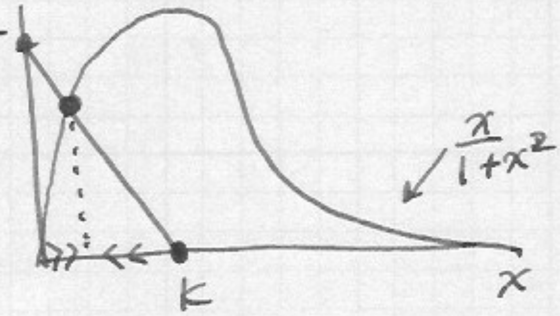
r = dimensionless growth rate

κ = dimensionless carrying capacity

6.5

Analysis of $\dot{x} = r x\left(1 - \frac{x}{K}\right) - \frac{x^2}{1+x^2}$ (Graphical)

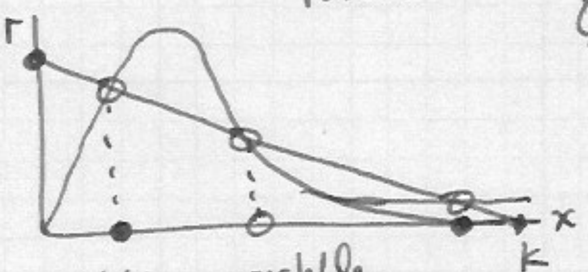
F.P. @ $r\left(1 - \frac{x}{K}\right) = \frac{x}{1+x^2}$ and $x^* = \phi$



One stable fixedpoint when ~~r~~ K is small

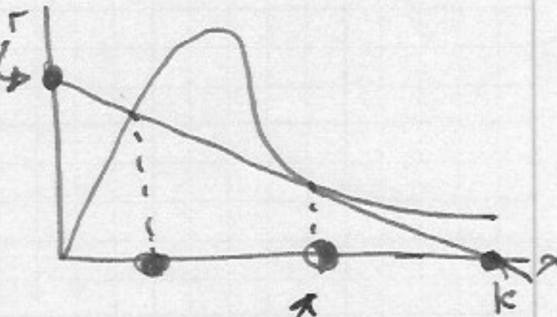
With large K: 3 Fixed Points

with lower r... a bifurcation



stable unstable

(a) (b) (c)



(a)

Saddle-node bifurcation

(annihilation of 2 F.P.)

Stability of fixed points:

-what is $x^* = \phi$?

$$\frac{\partial f(x)}{\partial x} = r - \frac{2rx}{K} - \overbrace{\frac{2x}{(1+x^2)^2}}^{\text{from quotient rule}}\Bigg|_{x^*=\phi} = r$$

~ unstable for positive r

(*) which means a → stable, b → unstable, c → stable. } b/c stability type must alternate

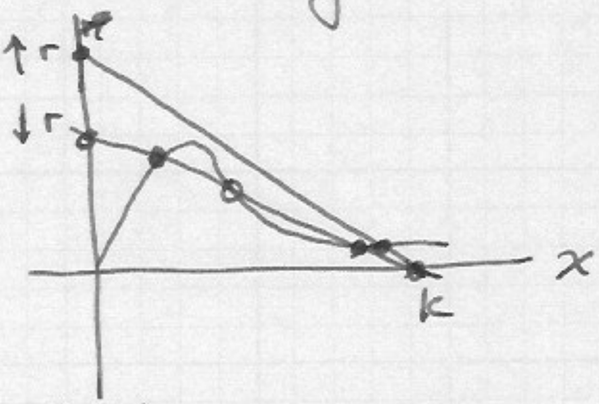
- Stable F.P. (a) functions as a refuge for budworms
- Stable F.P. (c) functions as the outbreak level

The fate of the system determined on x_0 ...

If $x_0 > (b)$, then ~~...~~ $x \to (c)$

So F.P. b is the <u>threshold</u>

Or if r increases .... F.P. (a) can disappear, and $x \to (c)$. if r is then lowered, the refuge F.P. will not immediately recover due to hysteresis.

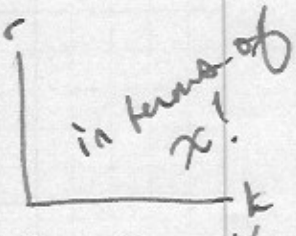


Calculating the bifurcations

↳ key parameters are k and r

(dimensionless carrying capacity

" growth rate



Condition: that the two lines become tangential... so that ~~their derivatives~~ are <u>equal</u>. the <u>lines</u> & <u>derivatives</u>

so: <u>the lines are equal</u>

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2}$$

the derivatives are equal

$$\frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right]$$

6.7

$$\frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right] = \frac{d}{dx}\left[\frac{x}{1+x^2}\right] \to -\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2}$$

expression for $\frac{r}{k}$

and we know:

$$r - \frac{rx}{k} = \frac{x}{1+x^2} \quad @ \text{ bifurcation}$$

substitute

$$r + \left(\frac{1-x^2}{(1+x^2)^2}\right)x = \frac{x}{1+x^2}$$

$$r = \frac{x}{1+x^2} - \frac{x(1-x^2)}{(1+x^2)^2} \quad \cancel{= \frac{2x^3}{(1+x^2)^2}}$$

$$\frac{x(1+x^2)}{(1+x^2)^2} - \frac{x(1-x^2)}{(1+x^2)^2} = \frac{\cancel{x}+x^3-\cancel{x}+x^3}{(1+x^2)^2}$$

$$\boxed{r = \frac{2x^3}{(1+x^2)^2}}$$ so we have r in terms of x

Now to get k in terms of x:

$$-\frac{r}{k} = \frac{1-x^2}{(1+x^2)^2} \to -\frac{2x^3}{(1+x^2)^2}\cdot\frac{1}{k} = \frac{1-x^2}{(1+x^2)^2}$$

$$\frac{-2x^3\cancel{(1+x^2)^2}}{\cancel{(1+x^2)^2}(1-x^2)} = k \to k = \frac{-2x^3}{1-x^2} \text{ or}$$

$$\boxed{k = \frac{2x^3}{x^2-1}}$$

also $k > 0$ so $x > 1$

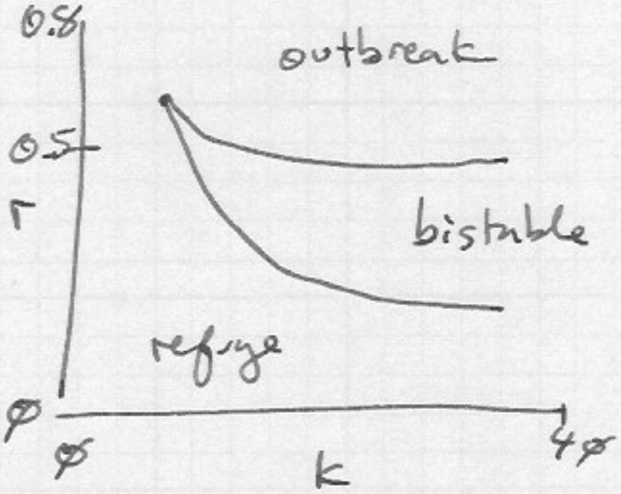
6.8

Plot bifurcation curves in k-r space

$$r = \frac{2x^3}{(1+x^2)^2} \quad \circledast \text{ for } x > 1$$

$$k = \frac{2x^3}{(x^2-1)}$$

↳ as we vary x, we get different (k, r) coordinates



What values are r & k for realistic forests?

$\circledast$ dimensionalize and apply knowledge of system:

Usually: $r < 0.5$ and $k \approx 300$.