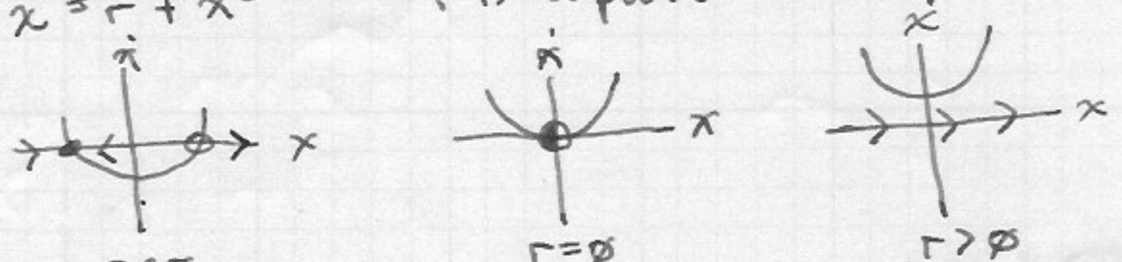
5.1 Bifurcations: Change in stability

Saddle-Node Bifurcations

- Mechanism by which bifurcations are created & destroyed
- A parameter is varied, 2 F.P. move towards each other, collide, and annihilate

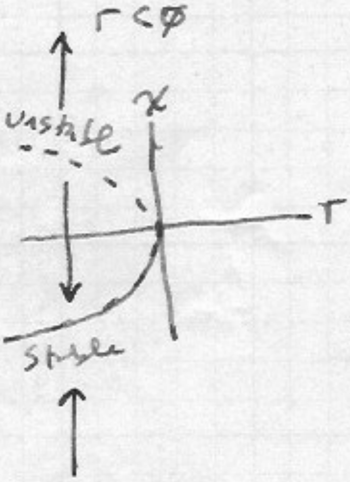
$\dot{x} = r + x^2$ r is a param



$r < \emptyset$ $r = \emptyset$ $r > \emptyset$

Bifurcation @ $r = \emptyset$

The interesting thing happens over 'r'! and x where $\dot{x}$ axis just follows where F.P. is.



vary r as a function of x

F.P. ~~x~~ $\emptyset = r + x^2$

~~x^2~~ $x^2 = -r$

$x^* = \pm\sqrt{-r}$

when $r < \emptyset$, sqrt is of a positive number

Ex) $\dot{x} = r - x^2$

F.P.: $x^* = \pm\sqrt{r}$ 2 F.P. for $r \geq \emptyset$

$\emptyset$ F.P. for $r < \emptyset$

$f'(x^*) =$ ~~$\frac{1}{2\sqrt{x^*}}$ $x^* = +\sqrt{r} \rightarrow -\frac{1}{2\sqrt{(\sqrt{r})}} = \frac{1}{2r^{1/4}}$~~

~~$\frac{1}{2\sqrt{x^*}}$ $x^* = -\sqrt{r} \rightarrow$~~

$= -2x^*$ $\rightarrow x^* \rightarrow +\sqrt{r} \rightarrow -2\sqrt{r}$ (stable)

$\rightarrow x^* \rightarrow -\sqrt{r} \rightarrow 2\sqrt{r}$ (unstable)

5.2 a $\dot{x} = r - x^2$

~~Ele~~

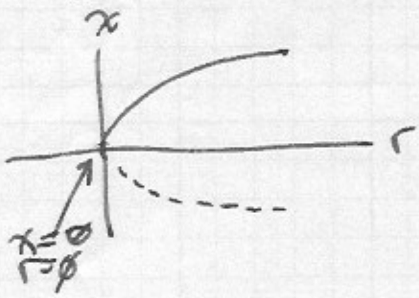
- We have found the F.P. @ $x^* = \pm\sqrt{r}$ for $r > \phi$
- We have found the stability ~~o~~ of F.P.:

$$\frac{\partial f(x=+\sqrt{r})}{\partial x} = -2\sqrt{r} \quad \text{(stable)}$$

$$\frac{\partial f(x=-\sqrt{r})}{\partial x} = 2\sqrt{r} \quad \text{(unstable)}$$

- What is the value of r where the behavior changes?

This is a simple example so we already know the answer:



above $r = \phi$ there are 2 F.P.

below $r = \phi$ there are NO F.P.

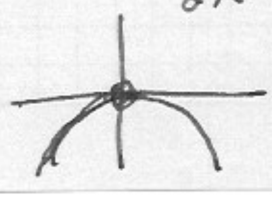
Fixed Points: so easy a kid could define it

BUT can they be SOLVED?

$\dot{x} = f(x)$ find F.P. by $\phi = f(x)$ + solve for x

What if we can't solve? Can we still say something about bifurcation?

- Where $\frac{\partial f(x)}{\partial x} = \phi$!

by 

Critical x^*

$$\frac{\partial}{\partial x}(r - x^2) = -2x$$

$$\phi = -2x_c^*$$

$$x_c^* = \phi$$

Critical r

And

$$\dot{x} = r - x^2$$

$$\phi = r - x^2$$

$$x^* = \pm\sqrt{r}$$

$$\phi = \pm\sqrt{r_c}$$

$$r_c = \phi$$

5.2b NO Analytical F.P., but we can still investigate bifurcation

Ex) $\dot{x} = r - x - e^{-x}$

1) break up into pieces

$(r-x)$ ~~two~~ ~line... how $\dot{x} \uparrow$

$-e^{-x}$ ~ exponential... how $\dot{x} \downarrow$

2) Graphically analyse it:

F.P. $r - x^* - e^{-x^*} = 0$

$\underbrace{r - x^* = e^{-x^*}}$ but can't find F.P. as a function of 'r'

VISUALLY, the bifurcation occurs when lines & tangents are equal

$f'(x)$ $\frac{\partial f(x)}{\partial x} = -1 + e^{-x}$ When $\frac{\partial}{\partial x}(r-x) = \frac{\partial}{\partial x} e^{-x}$

- The critical value of x occurs @ defined by $\frac{\partial f(x^*)}{\partial x} = 0$

- So: $-1 + e^{-x^*} = 0$

$e^{-x^*} = 1$

So: $x^* = 0$ ⊛ Bifurcation occurs @ $x = 0$

F.P. @ Bifurcation:

$r_c - x^* - e^{-x^*} = 0$

$r_c = 1$

§5.3 Transcritical Bifurcations

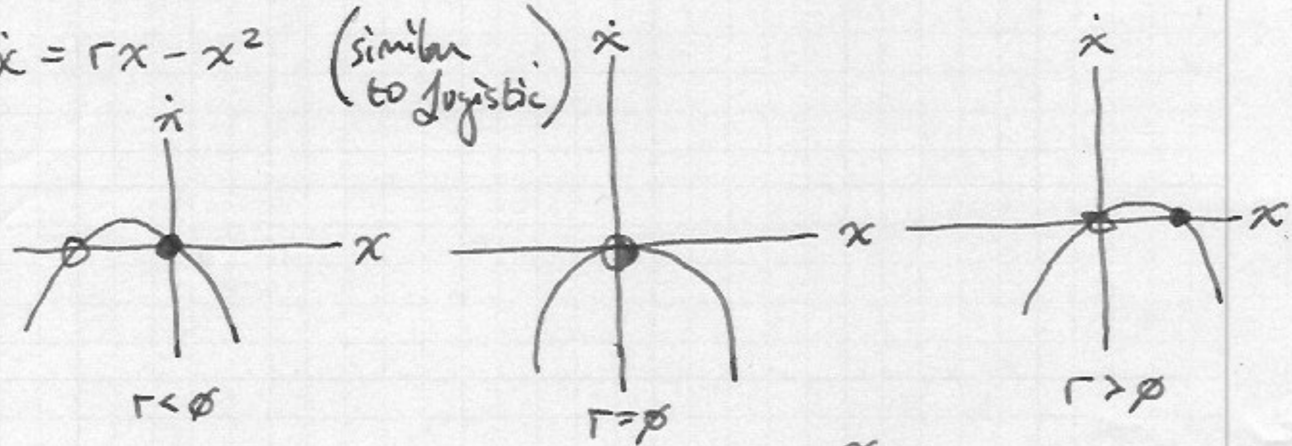
- Sometimes a ~~parameter~~ F.P. must exist across all values of a parameter and cannot be destroyed

- example: extinctions should always be a ~~parameter~~ F.P. in population dynamics

- But a F.P. can change its stability across a parameter!

- Normal Form

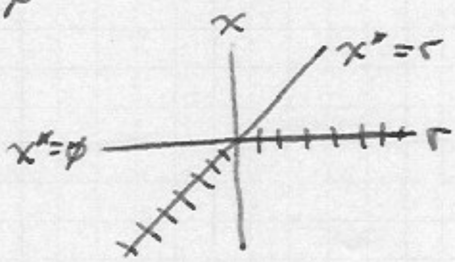
$\dot{x} = rx - x^2$ (similar to logistic)



F.P.: $\emptyset = rx - x^2 = x(r-x)$

$x^* = \emptyset$, $x^* = r$

(regardless of r)



Stability:

$x^*=\emptyset$ $\quad \left.\frac{d}{dx} rx - x^2 \right|_{x^*=\emptyset} = r - 2(\emptyset) = r$

so: when $r < \emptyset$, F.P. $x^* = \emptyset$ is stable

when $r > \emptyset$, F.P. $x^* = \emptyset$ is unstable

$x^*=r$ $\quad \left.\frac{d}{dx} rx - x^2 \right|_{x^*=r} = r - 2(r) = -r$

so: when $r < \emptyset$, F.P. $x^* = r$ is <u>unstable</u>

when $r > \emptyset$, F.P. $x^* = r$ is <u>stable</u>

⊛ There is an eXchange!

5.4

$$x - \overbrace{x^3}^{\text{very small}} - a(1-e^{-bx})$$

Show: $\dot{x} = x(1-x^2) - a(1-e^{-bx})$ undergoes TC bifurcation @ $x^* = \emptyset$

F.P. $\emptyset = x(1-x^2) - a(1-e^{-bx})$ ... $x^* = \emptyset$ is a fixed point.

An approximation is needed to get rid of e^{-bx}

Maclaurer expansion:

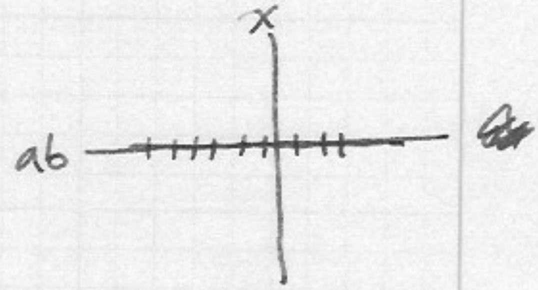
$$e^{bx} = 1 + \frac{1}{1!}bx + \frac{1}{2!}b^2x^2 + \underbrace{\frac{1}{3!}b^3x^3 \ldots \infty}$$

if x is small, we can ignore h.o.t.

$$e^{bx} \approx 1 + bx + \frac{b^2x^2}{2} \quad \text{for small } x$$

(and $x^* = \emptyset$ is small)

$$1 - e^{-bx} \approx 1 - \left[1 - bx + \frac{1}{2}b^2x^2\right]$$
$$= bx - \frac{1}{2}b^2x^2$$



so: ~~$\dot{x} \approx x - a(bx - \frac{1}{2}b^2x^2)$~~

$$\dot{x} = x - x^3 - a\left(bx - \frac{1}{2}b^2x^2 + \text{h.o.t.}\right)$$

very small too (includes x^3)

$$\approx x - a\left(bx - \frac{1}{2}b^2x^2\right) = x - abx + \frac{ab^2x^2}{2}$$

$$\dot{x} = x(1-ab) + \left(\frac{1}{2}ab^2\right)x^2$$

$$\frac{\partial f(x)}{\partial x} = (1-ab) + ab^2x \Big|_{x^*=\emptyset} = 1 - ab$$

if $ab > 1$, $x^* = \emptyset$ stable

if $ab < 1$, $x^* = \emptyset$ unstable

$\emptyset = 1 - ab$

Bifurcation @ $ab = 1$