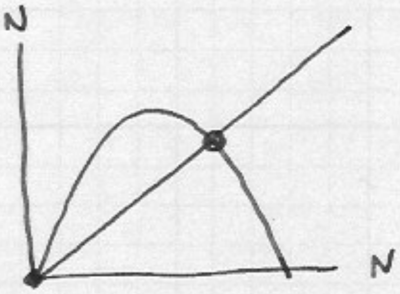
4.1

Adding more biological detail



$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - \mu N$$

$$\emptyset = rN\left(1 - \frac{N}{K}\right) - \mu N$$

$$rN\left(1 - \frac{N}{K}\right) = \mu N$$

$$\cancel{r - \frac{rN}{K} = \mu}$$

F.P.

$$1 - \frac{N}{K} = \frac{\mu}{r}$$

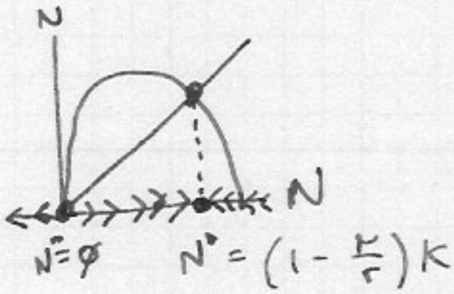
$$\frac{N}{K} = 1 - \frac{\mu}{r}$$

$$N^* = \left(1 - \frac{\mu}{r}\right)K$$ ~ when $\mu = \emptyset$, $N^* = K$ as before

Stability: when $rN\left(1 - \frac{N}{K}\right) > \mu N$, $\frac{dN}{dt} > \emptyset$ (growth)

when $rN\left(1 - \frac{N}{K}\right) < \mu N$, $\frac{dN}{dt} < \emptyset$ (decline)

So:



$N^* = \emptyset$ unstable

$N^* = \left(1 - \frac{\mu}{r}\right)K$ stable

Recall $\dot{N} = f(N)$

$\dot{\eta} = f'(N^*)\eta$ so: $\eta(t) = \eta(\emptyset)e^{f'(N^*)t}$

so: $f'(N^*)$ is the test function

if $f'(N^*) > \emptyset$ then N^* is unstable

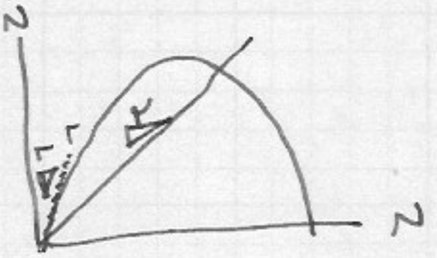
if $f'(N^*) < \emptyset$ then N^* is stable

4.2

$$f(N) = rN\left(1-\frac{N}{K}\right) - \mu N = rN - \frac{rN^2}{K} - \mu N$$

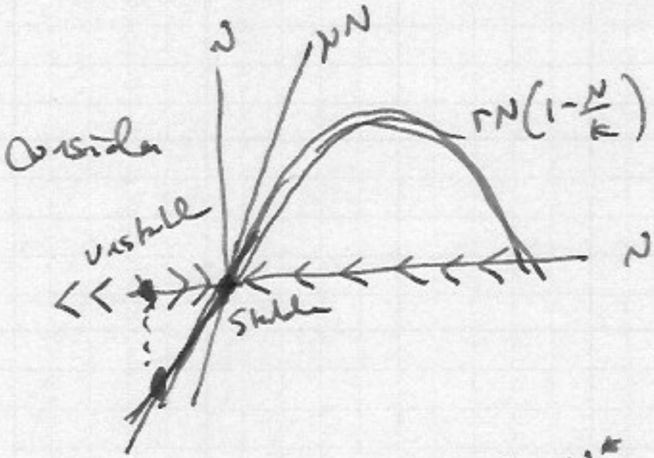
$$f'(N) = r - \frac{2rN}{K} - \mu$$

$$f'(N^* = \emptyset) = r - \emptyset - \mu = r - \mu \qquad (?)$$



$r \sim$ slope of parabola near $N = \emptyset$

$\mu \sim$ slope of mortality function



if $\mu > r$, then

$$N^* = \underbrace{\left(1 - \frac{\mu}{r}\right)}_{\text{Negative value}} K$$

$$f'\left(N^* = \left(1-\frac{\mu}{r}\right)K\right) = r - \frac{2rN^*}{K} - \mu$$

$$= r - \frac{2r\left[\left(1-\frac{\mu}{r}\right)\cancel{K}\right]}{\cancel{K}} - \mu$$

$$= r - 2r\left(1-\frac{\mu}{r}\right) - \mu$$

$$= r - 2r + \frac{2\cancel{r}\mu}{\cancel{r}} - \mu$$

$$= -r + \mu$$

$$= \mu - r$$

given we assume $\mu < r$, $f'(N^*)$ is $< \emptyset$ & stable.

Allee Effect

- growth rate is ~~more~~ negative @ low population densities
- Difficulty finding mates (mate limitation)
- Cooperative Defense: Increased energy devoted to predator-watching (less on mating) when N↓
  FOR low efficacy of anti-predator herding behavior ↑ predation rate
- Cooperative feeding (where foraging depends on pack-hunting for example)
- Engineering: where large populations can alter environment to advantage

Imagine critical N $N_c = A$ below which $\frac{dN}{dt} < \emptyset$ ...

then: $$\frac{dN}{dt} = rN\left(\frac{N}{A}-1\right)\left(1-\frac{N}{K}\right)$$

if $N < A$, $\left(\frac{N}{A}-1\right) < \emptyset$ and $\frac{dN}{dt} < \emptyset$

$$\emptyset = rN\left(\frac{N}{A}-1\right)\left(1-\frac{N}{K}\right)$$

$\underbrace{N^* = \emptyset \quad N^* = A \quad N^* = K}_{\text{F.P.}}$

4.4

F.P. $N^* = \{\varnothing, A, K\}$

$$\dot{N} = rN\left(\frac{N}{A} - 1\right)\left(1 - \frac{N}{K}\right)$$

$$\frac{\partial \dot{N}}{\partial N} = \left(\frac{rN^2}{A} - rN\right)\left(1 - \frac{N}{K}\right)$$

$$= \frac{rN^2}{A} - \frac{rN^3}{AK} - rN + \frac{rN^2}{K}$$

$$= \frac{2rN}{A} - \frac{3rN^2}{AK} - r + \frac{2rN}{K}$$

$$= \frac{r}{AK}\left[2(A+K)N - 3N^2 - AK\right]$$

for $N^* = \varnothing$, $\dfrac{\partial \dot{N}}{\partial N} = \dfrac{r}{AK}(-AK) = -r$ stable

for $N^* = A$, $\dfrac{\partial \dot{N}}{\partial N} = \dfrac{r}{AK}\left[2A^2 + 2\cancel{AK} - 3A^2 - \cancel{AK}\right]$

$$= \cancel{\frac{r}{AK}\left[-A^2\right]}$$

$$= \frac{r}{AK}\left[-A^2 + AK\right] = \frac{r}{K}\left[K - A\right] = r - \frac{Ar}{K}$$

if $\dfrac{A}{K} < 1$, then $r - \dfrac{A}{K}r > \varnothing$

unstable

for $N^* = K$ $\dfrac{\partial \dot{N}}{\partial N} = \dfrac{r}{AK}\left[2AK + 2K^2 - 3K^2 - AK\right]$

$$= \frac{r}{AK}\left[AK - K^2\right] = \frac{r}{A}\left[A - K\right] = r - \frac{rK}{A}$$

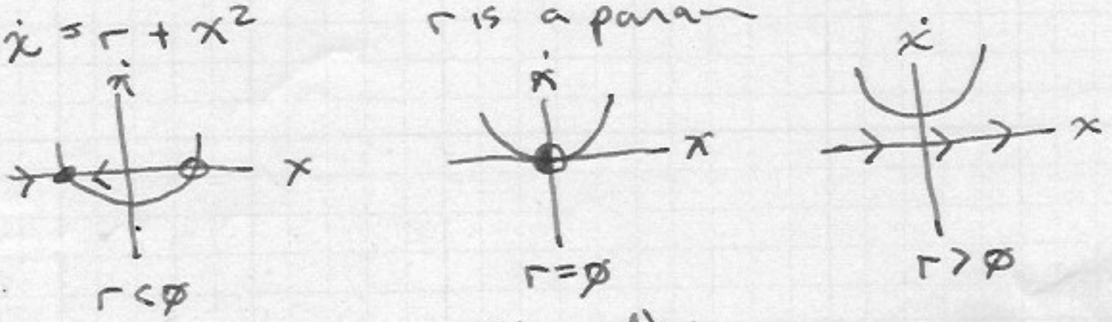
if $\dfrac{K}{A} > 1$, then $r - r\dfrac{K}{A} < \varnothing$ } Stable

5.1 Bifurcations: change in stability

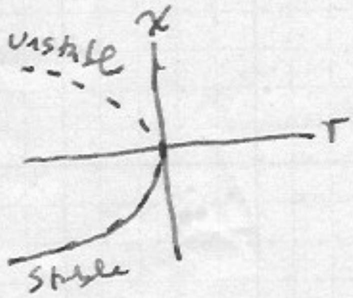
Saddle-Node Bifurcations

- Mechanism by which bifurcations are created & destroyed
- A parameter is varied, 2 F.P. move towards each other, collide, and annihilate

$\dot{x} = r + x^2$ r is a param



Bifurcation @ $r=\emptyset$



vary r as a function of x

F.P. $\emptyset = r + x^2$

$x^2 = -r$

$x^* = \pm\sqrt{-r}$

when $r<\emptyset$, sqrt is of a positive number

Ex) $\dot{x} = r - x^2$

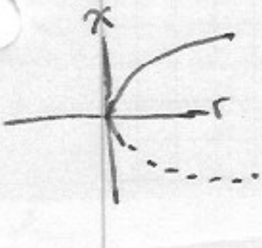
F.P.: $x^* = \pm\sqrt{r}$ 2 F.P. for $r \geq \emptyset$

$\emptyset$ F.P. for $r < \emptyset$

$f'(x^*) = -2x^*$

$x^* \to +\sqrt{r} \to -2\sqrt{r}$ (stable)

$x^* \to -\sqrt{r} \to 2\sqrt{r}$ (unstable)



5.2

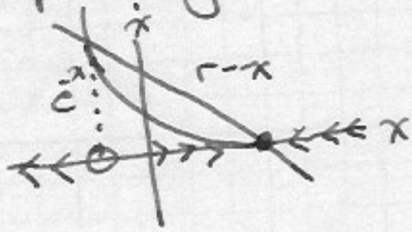
Ex) $\dot{x} = r - x - e^{-x}$

1) break up into pieces

$(r-x)$ ~~tee~~ ~line... how $\dot{x} \uparrow$

$-e^{-x}$ ~ exponential... how $\dot{x} \downarrow$

2) Graphically analyse it:

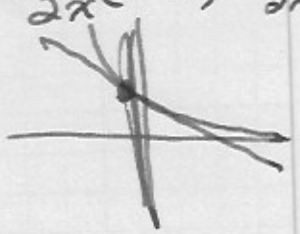


F.P. $r - x^* - e^{-x^*} = \emptyset$

$r - x^* = e^{-x^*}$ but can't find F.P. as a function of 'r'

Visually, the bifurcation occurs when lines & tangents are eq-

$\cancel{f'(x)}$ $\frac{\partial f(x)}{\partial x} = \cancel{\quad} -1 + e^{-x}$ When $\frac{\partial}{\partial x}(r-x) = \frac{\partial}{\partial x}e^{-x}$



- The critical value of x occurs @ defined by $\frac{\partial f(x^*)}{\partial x} = \emptyset$

- So: $-1 + e^{-x^*} = \emptyset$

$e^{-x^*} = 1$

so: $x^* = \emptyset$ ⊛ Bifurcation occurs @ $x = \emptyset$

F.P. @ Bifurcation:

$r_c - x^* - e^{-x^*} = \emptyset$

$r_c = 1$