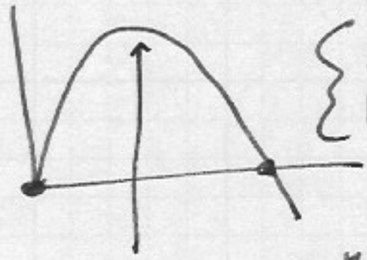
3.1

$$\frac{dN}{dt} = rN \quad \text{vs.} \quad \frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) = rN - \frac{rN^2}{K}$$

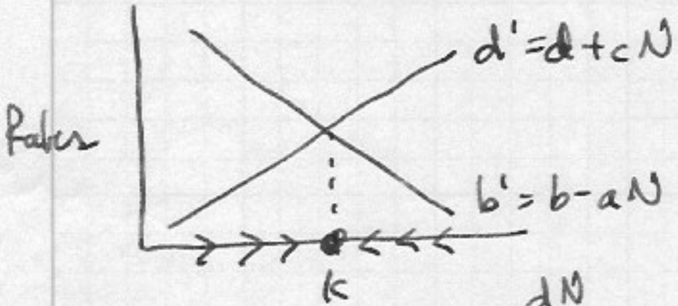
What are the Fixed Points? Steady states? $\cancel{\emptyset = rN - \frac{rN^2}{K}}$

$$\emptyset = \frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

~independent of r



$$\begin{cases} N^* = \emptyset \\ N^* = K \end{cases}$$



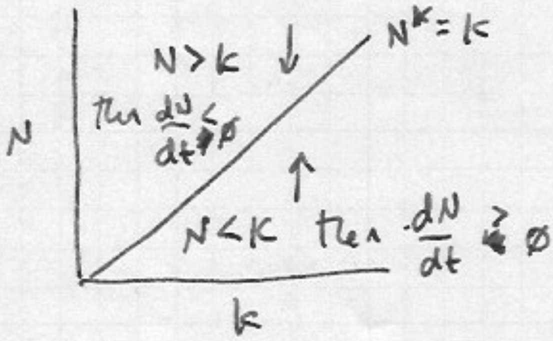
$$\frac{dN}{dt} = \left[b' - d'\right]N$$

$$\to \left[(b-aN)-(d+cN)\right]N$$

$$\frac{dN}{dt} = rN\left[1 - \frac{(a+c)}{(b-d)}N\right] \to rN\left[1-\frac{N}{K}\right]$$

Define $K = \frac{(b-d)}{(a+c)}$

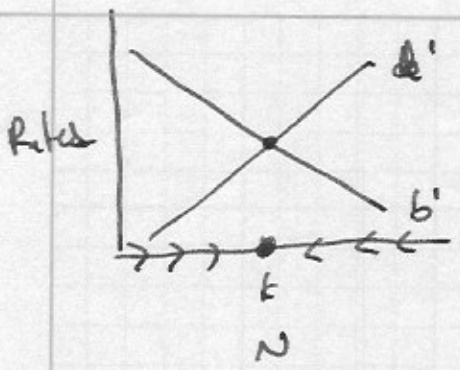
↳ the point where $b' = d'$



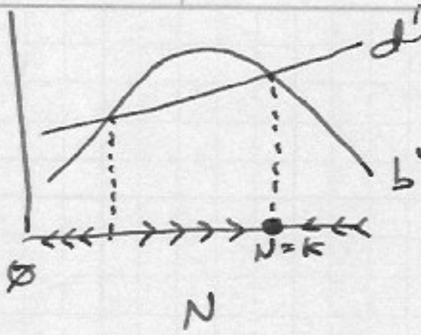
- Small perturbation off $N^* = K$ .... what happens?

= $N^* = K$ IS STABLE

3.2



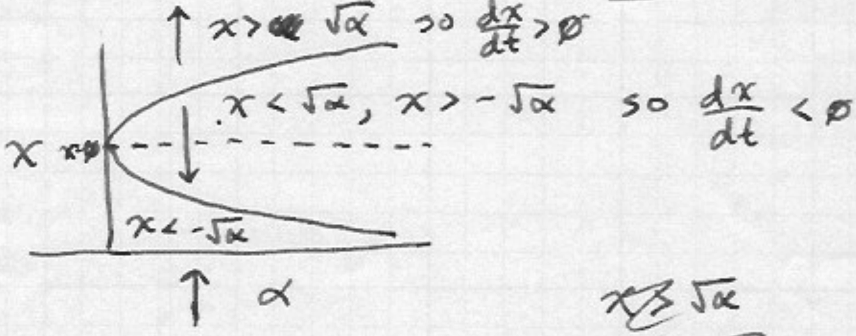
vs.

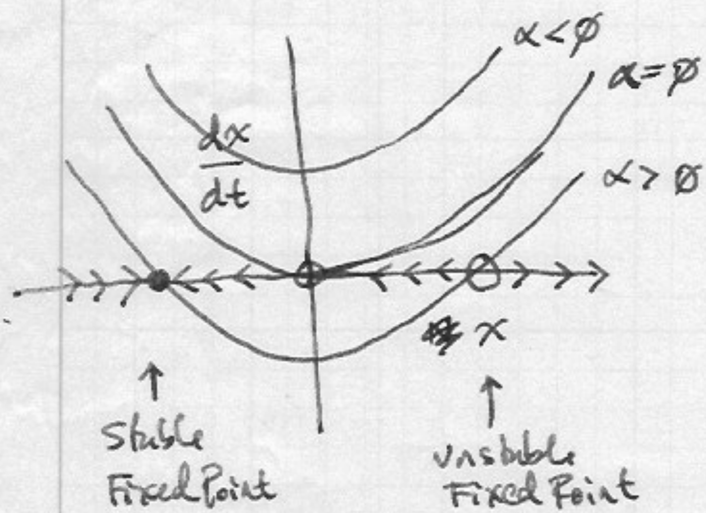


- lower birth rates at very low density Allee effect

Ex) $\frac{dx}{dt} = x^2 - \alpha$

$\emptyset = x^2 - \alpha \rightarrow \alpha = x^2 \rightarrow \boxed{\pm\sqrt{\alpha} = x^*}$





Saddle-node bifurcation!

~~$x \geq \sqrt{\alpha}$~~

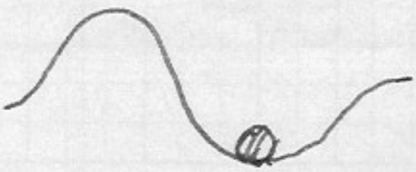
~~so: $\frac{dx}{dt} = (\sqrt{\alpha})^2 - \alpha$~~

@ $x = \sqrt{\alpha}$

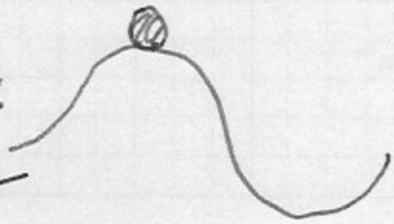
$$\frac{dx}{dt} = (\sqrt{\alpha})^2 - \alpha = \alpha - \alpha = \emptyset$$

@ $\alpha = \emptyset$, the system is structurally unstable... small changes in α lead to
- No fixed points
- 2 fixed points.

Stable FP:
Attractor
Sink



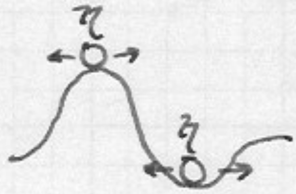
Unstable FP:
Repellors
Sources



3.3

- We have a system given by $\overset{\dot{x}}{\frac{dx}{dt}} = f(x)$
- We can find fixed points by setting $\dot{x} = \varnothing$
- We can see from previous examples that the quality of the f.p. is crucial for determining system dynamics.

We want to know what happens when a system is pushed near its fixed point:



~~what with happ~~

What are the dynamics of a $\overset{\text{small}}{\text{perturbation}}$ η near the fixed point

~ We can phrase the question in terms of perturbation dynamics...

- <u>Unstable</u>: ~~fre~~ perturbation <u>Grows</u>! $\frac{d\eta}{dt} > \varnothing$
- <u>Stable</u>: perturbation Declines! $\frac{d\eta}{dt} < \varnothing$

Say $\dot{x} = f(x)$ is the system with F.P. x^*

Let $\eta(t) = x(t) - x^*$ ~ small perturbation away from x^*

Q: Does the perturbation grow or decline?

$$\dot{\eta} = \frac{d}{dt}\left(x(t) - x^*\right) = \frac{dx(t)}{dt} - \varnothing \quad \text{b/c } x^* \text{ is constant}$$

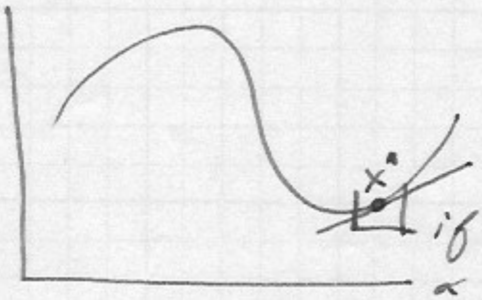
$$\dot{\eta} = \dot{x}$$

$$\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta) \quad \text{b/c } x(t) = x^* + \eta$$

3.4

$\dot{\eta} = \dot{x} = f(x) = \underbrace{f(x^* + \eta)}$

Could be a complicated, unknown function... but we only care about what happens in the vicinity of x^*...



if we zoom in enough, a linear approximation is enough...

Linear approximation via Taylor Expansion.

$$f(x) \text{ near } a \doteq f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

~~if ... is small~~

$f(x(t))$ near x^* where $x(t) = x^* + \eta$

$$f(x^* + \eta) = f(x^*) + \frac{f'(x^*)}{1!}(x - x^*) + \frac{f''(x^*)}{2!}(x-x^*)^2$$

$$\cancel{\approx f(x^*) + f'(x^*)(\eta)}$$

$$= f(x^*) + \eta f'(x) + \eta^2 \frac{f''(x^*)}{2!} + \dots \quad b/c \ \eta = x(t) - x^*$$

Now... recall η is VERY SMALL .... s.t. η^2 is VERYVERY small. Ignore it.

3.5

so:

$$f(x(t)) = f(x^* + \eta) \approx \underbrace{f(x^*)} + \eta f'(x^*)$$

$$f(x^*) = \frac{dx}{dt} @ x^* = \underline{\underline{\emptyset}}$$

$\dot{\eta} \approx \eta f'(x^*)$ this is the linearization about x^*

What does $\eta(t)$ look like?

$\dot{\eta} = f'(x^*)\eta \rightarrow \eta(t) \doteq \eta e^{f'(x^*)t}$

looks familiar...

$\dot{N} = rN \ldots$

if $r > \emptyset$, exponential growth

if $r < \emptyset$, exponential decay

$\cancel{\eta(t) = f'(x^*) \log t(\eta)}$

Same with $\dot{\eta} = f'(x^*)\eta$

if $f'(x^*) = \left.\frac{\partial f}{\partial x}\right|_{x^*} > \emptyset$, $\eta \uparrow$ UNSTABLE

if $f'(x^*) = \left.\frac{\partial f}{\partial x}\right|_{x^*} < \emptyset$, $\eta \downarrow$ STABLE

Ex) The Logistic Eqn:

$\dot{N} = rN\left(1 - \frac{N}{K}\right)$

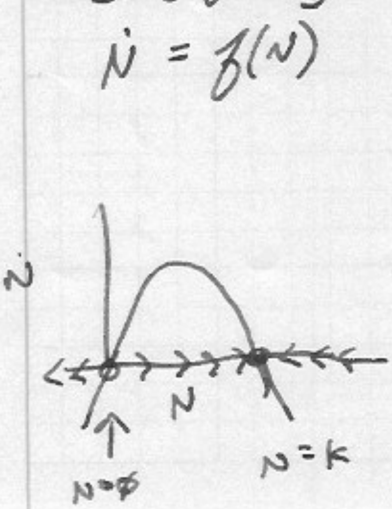
[$\cancel{\dot{x} = f(x)}$]

$\dot{N} = f(N)$

here, $\cancel{\dot{x}}$ f(N)

$$f(N) = rN - \frac{rN^2}{K}$$

$$\frac{d f(N)}{dN} = r - \frac{2rN}{K}$$



$$\left.\frac{\partial f(N)}{\partial N}\right|_{N^*} = r - \frac{2rK}{K} = r - 2r = -r < \emptyset$$

$$r - \frac{2r(\emptyset)}{K} = r \text{ UNSTABLE}$$

3.6

(+, -)

Slope of $f'(x^*)$ @ fixed point determines stability

~ But <u>How</u> stable? Magnitude of $f'(x^*)$

Characteristic timescale of the system

- The time required for $x(t)$ to vary significantly in the neighborhood of x^*

$$= \frac{1}{|f'(x^*)|}$$

Given that for the logistic eqn, $f'(\emptyset) = r$ unstable

$f'(K) = -r$ stable

Timescale $= \frac{1}{r}$

High growth rate $\rightarrow$ short timescale

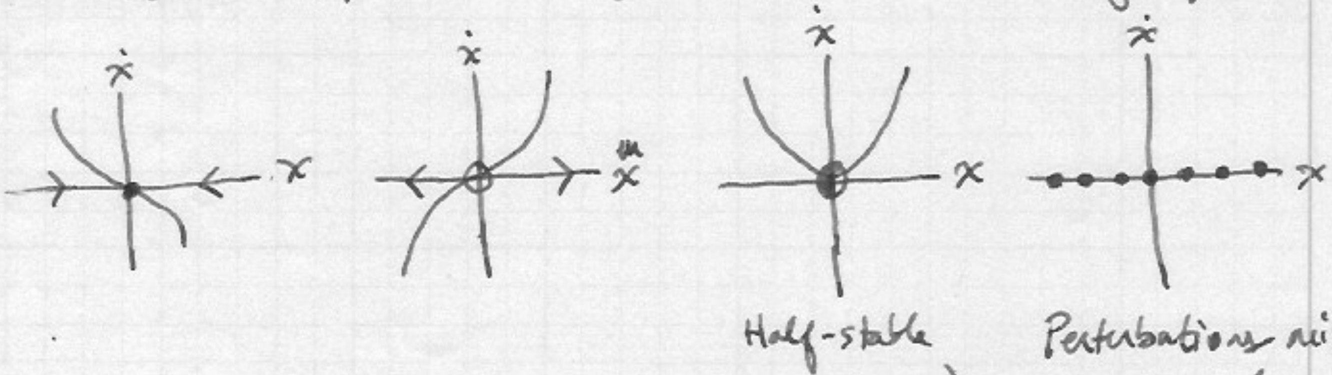
<u>Nonlinear</u> stability Analysis

$f'(x^*) = \emptyset$?

- Sol'n: Nothing can be said except on case-by-case basis.

(a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = \emptyset$

In each case, $x^* = \emptyset$ & $f'(x^*) = \emptyset$ but stability different



Half-stable (Indeterminant)

Perturbations neither grow or decay!