3

BIFURCATIONS

3.0 Introduction

As we've seen in Chapter 2, the dynamics of vector fields on the line is very limited: all solutions either settle down to equilibrium or head out to $\pm\infty$. Given the triviality of the dynamics, what's interesting about one-dimensional systems? Answer: *Dependence on parameters.* The qualitative structure of the flow can change as parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called ***bifurcations,*** and the parameter values at which they occur are called ***bifurcation points.***

Bifurcations are important scientifically—they provide models of transitions and instabilities as some *control parameter* is varied. For example, consider the buckling of a beam. If a small weight is placed on top of the beam in Figure 3.0.1, the beam can support the load and remain vertical. But if the load is too heavy, the vertical position becomes unstable, and the beam may buckle.

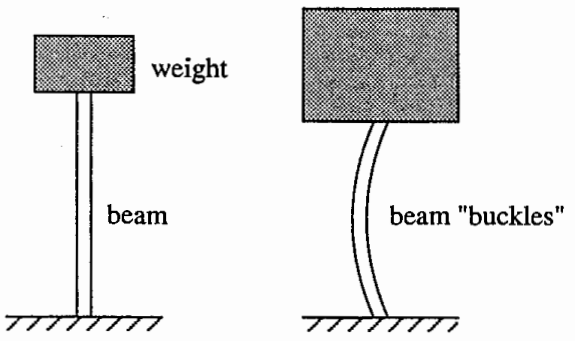


Figure 3.0.1

Here the weight plays the role of the control parameter, and the deflection of the beam from vertical plays the role of the dynamical variable x.

One of the main goals of this book is to help you develop a solid and practical understanding of bifurcations. This chapter introduces the simplest examples: bifurcations of fixed points for flows on the line. We'll use these bifurcations to model such dramatic phenomena as the onset of coherent radiation in a laser and the outbreak of an insect population. (In later chapters, when we step up to two- and three-dimensional phase spaces, we'll explore additional types of bifurcations and their scientific applications.)

We begin with the most fundamental bifurcation of all.

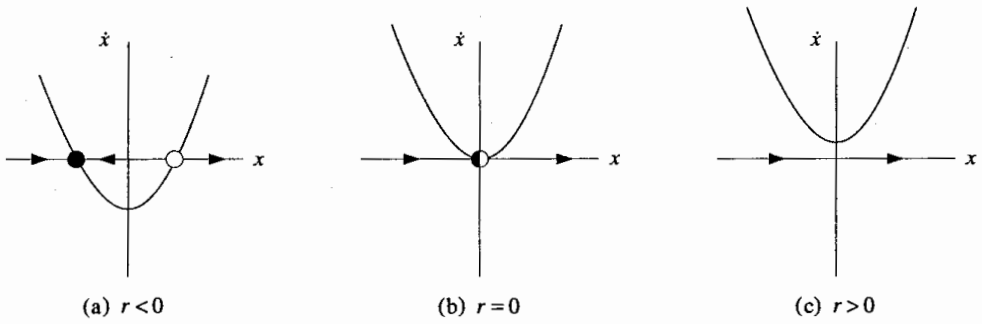
3.1 Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed.* As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first-order system

$$\dot{x} = r + x^2 \qquad (1)$$

where r is a parameter, which may be positive, negative, or zero. When r is negative, there are two fixed points, one stable and one unstable (Figure 3.1.1a).



(a) $r<0$ (b) $r=0$ (c) $r>0$

Figure 3.1.1

As r approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x^* = 0$ (Figure 3.1.1b). This type of fixed point is extremely delicate—it vanishes as soon as $r > 0$, and now there are no fixed points at all (Figure 3.1.1c).

In this example, we say that a *bifurcation* occurred at $r = 0$, since the vector fields for $r < 0$ and $r > 0$ are qualitatively different.

Graphical Conventions

There are several other ways to depict a saddle-node bifurcation. We can show a stack of vector fields for discrete values of r (Figure 3.1.2).

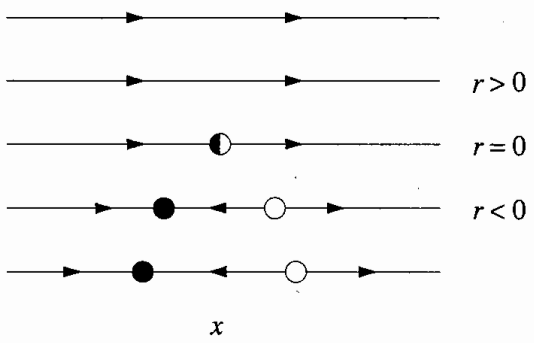


Figure 3.1.2

This representation emphasizes the dependence of the fixed points on r. In the limit of a *continuous* stack of vector fields, we have a picture like Figure 3.1.3. The curve shown is $r = -x^2$, i.e., $\dot{x} = 0$, which gives the fixed points for different r. To distinguish between stable and unstable fixed points, we use a solid line for stable points and a broken line for unstable ones.

However, the most common way to depict the bifurcation is to invert the axes of Figure 3.1.3. The rationale is that r plays the role of an independent variable, and so should be plotted horizontally

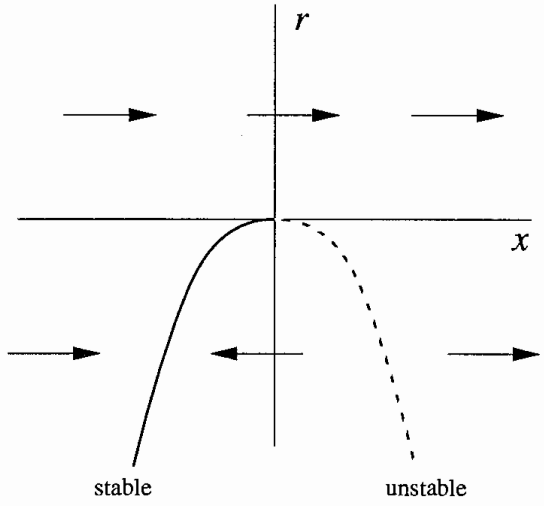


Figure 3.1.3

(Figure 3.1.4). The drawback is that now the x-axis has to be plotted vertically, which looks strange at first. Arrows are sometimes included in the picture, but not always. This picture is called the ***bifurcation diagram*** for the saddle-node bifurcation.

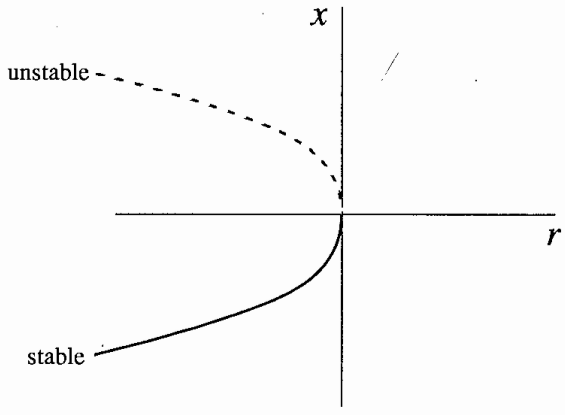


Figure 3.1.4

Terminology

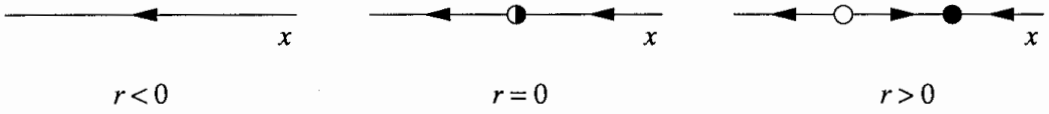
Bifurcation theory is rife with conflicting terminology. The subject really hasn't settled down yet, and different people use different words for the same thing. For example, the saddle-node bifurcation

is sometimes called a *fold bifurcation* (because the curve in Figure 3.1.4 has a fold in it) or a *turning-point bifurcation* (because the point $(x,r)=(0,0)$ is a "turning point.") Admittedly, the term *saddle-node* doesn't make much sense for vector fields on the line. The name derives from a completely analogous bifurcation seen in a higher-dimensional context, such as vector fields on the plane, where fixed points known as saddles and nodes can collide and annihilate (see Section 8.1).

The prize for most inventive terminology must go to Abraham and Shaw (1988), who write of a ***blue sky bifurcation***. This term comes from viewing a saddle-node bifurcation in the other direction: a pair of fixed points appears "out of the clear blue sky" as a parameter is varied. For example, the vector field

$$\dot{x} = r - x^2 \tag{2}$$

has no fixed points for $r<0$, but then one materializes when $r=0$ and splits into two when $r>0$ (Figure 3.1.5). Incidentally, this example also explains why we use the word "bifurcation": it means "splitting into two branches."



$r<0$ $\qquad$ $r=0$ $\qquad$ $r>0$

Figure 3.1.5

EXAMPLE 3.1.1:

Give a linear stability analysis of the fixed points in Figure 3.1.5.

Solution: The fixed points for $\dot{x}=f(x)=r-x^2$ are given by $x^*=\pm\sqrt{r}$. There are two fixed points for $r\geq 0$, and none for $r<0$. To determine linear stability, we compute $f'(x^*)=-2x^*$. Thus $x^*=+\sqrt{r}$ is stable, since $f'(x^*)<0$. Similarly $x^*=-\sqrt{r}$ is unstable. At the bifurcation point $r=0$, we find $f'(x^*)=0$; the linearization vanishes when the fixed points coalesce. ■

EXAMPLE 3.1.2:

Show that the first-order system $\dot{x}=r-x-e^{-x}$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point.

Solution: The fixed points satisfy $f(x)=r-x-e^{-x}=0$. But now we run into a difficulty—in contrast to Example 3.1.1, we can't find the fixed points explicitly as a function of r. Instead we adopt a geometric approach. One method would be to graph the function $f(x)=r-x-e^{-x}$ for different values of r, look for its roots x^*, and then sketch the vector field on the x-axis. This method is

fine, but there's an easier way. The point is that the two functions $r-x$ and e^{-x} have much more familiar graphs than their difference $r-x-e^{-x}$. So we plot $r-x$ and e^{-x} on the same picture (Figure 3.1.6a). Where the line $r-x$ intersects the curve e^{-x}, we have $r-x=e^{-x}$ and so $f(x)=0$. *Thus, intersections of the line and the curve correspond to fixed points for the system.* This picture also allows us to read off the direction of flow on the x-axis: the flow is to the right where the line lies above the curve, since $r-x>e^{-x}$ and therefore $\dot{x}>0$. Hence, the fixed point on the right is stable, and the one on the left is unstable.

Now imagine we start decreasing the parameter r. The line $r-x$ slides down and the fixed points approach each other. At some critical value $r=r_c$, the line becomes *tangent* to the curve and the fixed points coalesce in a saddle-node bifurcation (Figure 3.1.6b). For r below this critical value, the line lies below the curve and there are no fixed points (Figure 3.1.6c).

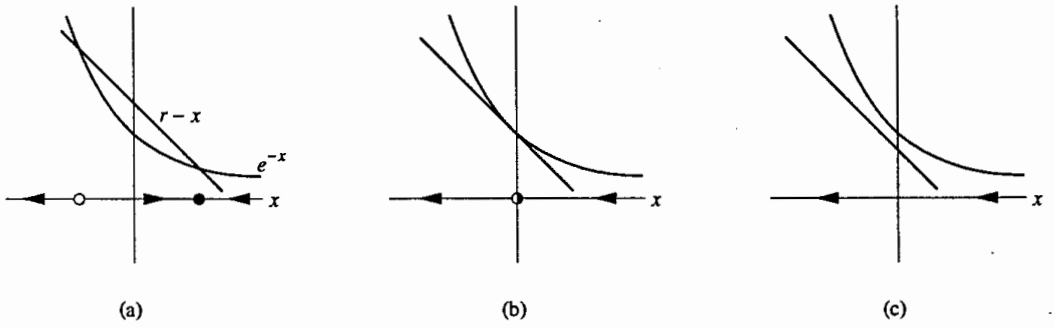


Figure 3.1.6

To find the bifurcation point r_c, we impose the condition that the graphs of $r-x$ and e^{-x} intersect *tangentially*. Thus we demand equality of the functions *and* their derivatives:

$$e^{-x}=r-x$$

and

$$\tfrac{d}{dx}e^{-x}=\tfrac{d}{dx}(r-x).$$

The second equation implies $-e^{-x}=-1$, so $x=0$. Then the first equation yields $r=1$. Hence the bifurcation point is $r_c=1$, and the bifurcation occurs at $x=0$. ■

Normal Forms

In a certain sense, the examples $\dot{x}=r-x^2$ or $\dot{x}=r+x^2$ are representative of *all* saddle-node bifurcations; that's why we called them "prototypical." The idea is that, close to a saddle-node bifurcation, the dynamics typically look like $\dot{x}=r-x^2$ or $\dot{x}=r+x^2$.

For instance, consider Example 3.1.2 near the bifurcation at $x = 0$ and $r = 1$. Using the Taylor expansion for e^{-x} about $x = 0$, we find

$$\begin{aligned}\dot{x} &= r - x - e^{-x} \\ &= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right] \\ &= (r-1) - \frac{x^2}{2} + \cdots\end{aligned}$$

to leading order in x. This has the same algebraic form as $\dot{x} = r - x^2$, and can be made to agree exactly by appropriate rescalings of x and r.

It's easy to understand why saddle-node bifurcations typically have this algebraic form. We just ask ourselves: how can two fixed points of $\dot{x} = f(x)$ collide and disappear as a parameter r is varied? Graphically, fixed points occur where the graph of $f(x)$ intersects the x-axis. For a saddle-node bifurcation to be possible, we need two nearby roots of $f(x)$; this means $f(x)$ must look locally "bowl-shaped" or parabolic (Figure 3.1.7).

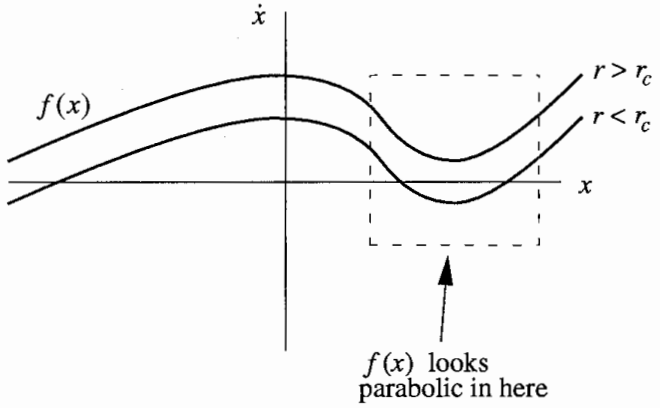


Figure 3.1.7

Now we use a microscope to zoom in on the behavior near the bifurcation. As r varies, we see a parabola intersecting the x-axis, then becoming tangent to it, and then failing to intersect. This is exactly the scenario in the prototypical Figure 3.1.1.

Here's a more algebraic version of the same argument. We regard f as a function of both x and r, and examine the behavior of $\dot{x} = f(x,r)$ near the bifurcation at $x = x^*$ and $r = r_c$. Taylor's expansion yields

$$\begin{aligned}\dot{x} &= f(x,r) \\ &= f(x^*, r_c) + (x - x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots\end{aligned}$$

where we have neglected quadratic terms in $(r - r_c)$ and cubic terms in $(x - x^*)$. Two of the terms in this equation vanish: $f(x^*, r_c) = 0$ since x^* is a fixed point, and $\partial f/\partial x|_{(x^*, r_c)} = 0$ by the tangency condition of a saddle-node bifurcation. Thus

$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \cdots \qquad (3)$$

where $a = \partial f/\partial r|_{(x^*, r_c)}$ and $b = \frac{1}{2}\partial^2 f/\partial x^2|_{(x^*, r_c)}$. Equation (3) agrees with the form of our prototypical examples. (We are assuming that $a, b \neq 0$, which is the typical case; for instance, it would be a very special situation if the second derivative $\partial^2 f/\partial x^2$ also happened to vanish at the fixed point.)

What we have been calling prototypical examples are more conventionally known as ***normal forms*** for the saddle-node bifurcation. There is much, much more to normal forms than we have indicated here. We will be seeing their importance throughout this book. For a more detailed and precise discussion, see Guckenheimer and Holmes (1983) or Wiggins (1990).

3.2 Transcritical Bifurcation

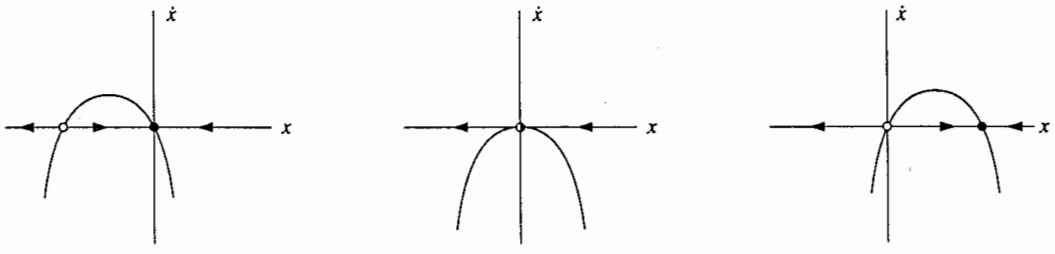
There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed. For example, in the logistic equation and other simple models for the growth of a single species, there is a fixed point at zero population, regardless of the value of the growth rate. However, such a fixed point may *change its stability* as the parameter is varied. The transcritical bifurcation is the standard mechanism for such changes in stability.

The normal form for a transcritical bifurcation is

$$\dot{x} = rx - x^2. \qquad (1)$$

This looks like the logistic equation of Section 2.3, but now we allow x and r to be either positive or negative.

Figure 3.2.1 shows the vector field as r varies. Note that there is a fixed point at $x^* = 0$ for *all* values of r.



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$

Figure 3.2.1

For $r<0$, there is an unstable fixed point at $x^*=r$ and a stable fixed point at $x^*=0$. As r increases, the unstable fixed point approaches the origin, and coalesces with it when $r=0$. Finally, when $r>0$, the origin has become unstable, and $x^*=r$ is now stable. Some people say that an ***exchange of stabilities*** has taken place between the two fixed points.

Please note the important difference between the saddle-node and transcritical bifurcations: in the transcritical case, the two fixed points don't disappear after the bifurcation—instead they just switch their stability.

Figure 3.2.2 shows the bifurcation diagram for the transcritical bifurcation. As in Figure 3.1.4, the parameter r is regarded as the independent variable, and the fixed points $x^*=0$ and $x^*=r$ are shown as dependent variables.

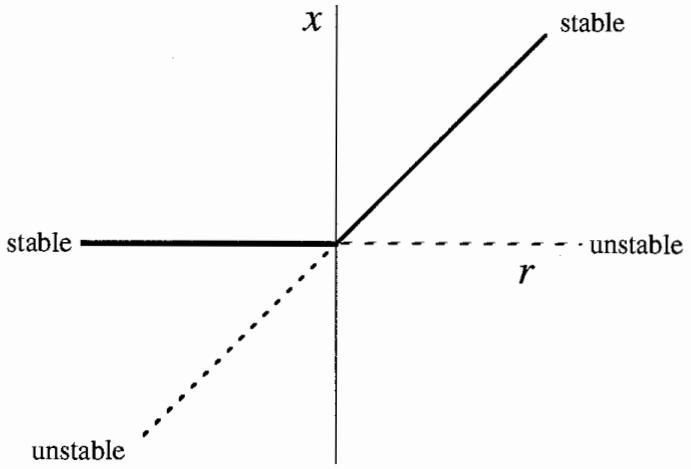


Figure 3.2.2

EXAMPLE 3.2.1:

Show that the first-order system $\dot{x}=x(1-x^2)-a(1-e^{-bx})$ undergoes a transcritical bifurcation at $x=0$ when the parameters a, b satisfy a certain equation, to be determined. (This equation defines a ***bifurcation curve*** in the (a,b) parameter space.) Then find an approximate formula for the fixed point that bifurcates from $x=0$, assuming that the parameters are close to the bifurcation curve.

Solution: Note that $x=0$ is a fixed point for all (a,b). This makes it plausible that the fixed point will bifurcate transcritically, if it bifurcates at all. For small x, we find

$$\begin{aligned} 1-e^{-bx} &= 1-\left[1-bx+\tfrac{1}{2}b^2x^2+O(x^3)\right] \\ &= bx-\tfrac{1}{2}b^2x^2+O(x^3) \end{aligned}$$

and so

$$\begin{aligned} \dot{x} &= x-a(bx-\tfrac{1}{2}b^2x^2)+O(x^3) \\ &= (1-ab)x+(\tfrac{1}{2}ab^2)x^2+O(x^3). \end{aligned}$$

Hence a transcritical bifurcation occurs when $ab = 1$; this is the equation for the bifurcation curve. The nonzero fixed point is given by the solution of $1 - ab + (\frac{1}{2}ab^2)x \approx 0$, i.e.,

$$x^* \approx \frac{2(ab-1)}{ab^2}.$$

This formula is approximately correct only if x^* is small, since our series expansions are based on the assumption of small x. Thus the formula holds only when ab is close to 1, which means that the parameters must be close to the bifurcation curve. ■

EXAMPLE 3.2.2:

Analyze the dynamics of $\dot{x} = r\ln x + x - 1$ near $x = 1$, and show that the system undergoes a transcritical bifurcation at a certain value of r. Then find new variables X and R such that the system reduces to the approximate normal form $\dot{X} \approx RX - X^2$ near the bifurcation.

Solution: First note that $x = 1$ is a fixed point for all values of r. Since we are interested in the dynamics near this fixed point, we introduce a new variable $u = x - 1$, where u is small. Then

$$\begin{aligned}\dot{u} &= \dot{x} \\ &= r\ln(1+u) + u \\ &= r\left[u - \tfrac{1}{2}u^2 + O(u^3)\right] + u \\ &\approx (r+1)u - \tfrac{1}{2}ru^2 + O(u^3).\end{aligned}$$

Hence a transcritical bifurcation occurs at $r_c = -1$.

To put this equation into normal form, we first need to get rid of the coefficient of u^2. Let $u = av$, where a will be chosen later. Then the equation for v is

$$\dot{v} = (r+1)v - (\tfrac{1}{2}ra)v^2 + O(v^3).$$

So if we choose $a = 2/r$, the equation becomes

$$\dot{v} = (r+1)v - v^2 + O(v^3).$$

Now if we let $R = r + 1$ and $X = v$, we have achieved the approximate normal form $\dot{X} \approx RX - X^2$, where cubic terms of order $O(X^3)$ have been neglected. In terms of the original variables, $X = v = u/a = \frac{1}{2}r(x-1)$. ■

To be a bit more accurate, the theory of normal forms assures us that we can find a change of variables such that the system becomes $\dot{X} = RX - X^2$, with *strict,* rather than approximate, equality. Our solution above gives an approximation to the necessary change of variables. If we wanted a better approximation, we would

retain the cubic terms in the series expansions (and perhaps even higher-order terms if we're really feeling heroic) and we would have to do a more elaborate calculation to eliminate these higher-order terms. See Exercises 3.2.6 and 3.2.7 for a taste of such calculations, or see the books of Guckenheimer and Holmes (1983), Wiggins (1990), or Manneville (1990).

3.3 Laser Threshold

Now it's time to apply our mathematics to a scientific example. We analyze an extremely simplified model for a laser, following the treatment given by Haken (1983).

Physical Background

We are going to consider a particular type of laser known as a solid-state laser, which consists of a collection of special "laser-active" atoms embedded in a solid-state matrix, bounded by partially reflecting mirrors at either end. An external energy source is used to excite or "pump" the atoms out of their ground states (Figure 3.3.1).

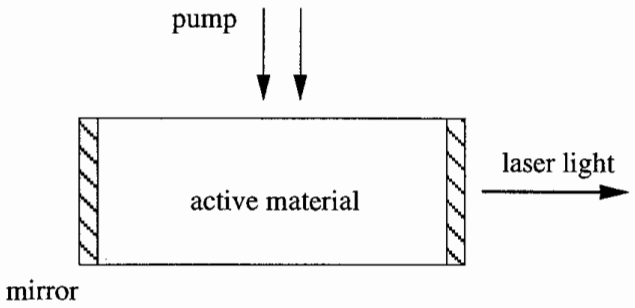


Figure 3.3.1

Each atom can be thought of as a little antenna radiating energy. When the pumping is relatively weak, the laser acts just like an ordinary *lamp*: the excited atoms oscillate independently of one another and emit randomly phased light waves.

Now suppose we increase the strength of the pumping. At first nothing different happens, but then suddenly, when the pump strength exceeds a certain threshold, the atoms begin to oscillate in phase—the lamp has turned into a ***laser***. Now the trillions of little antennas act like one giant antenna and produce a beam of radiation that is much more coherent and intense than that produced below the laser threshold.

This sudden onset of coherence is amazing, considering that the atoms are being excited completely at random by the pump! Hence the process is *self-organizing*: the coherence develops because of a cooperative interaction among the atoms themselves.

Model

A proper explanation of the laser phenomenon would require us to delve into quantum mechanics. See Milonni and Eberly (1988) for an intuitive discussion.

Instead we consider a simplified model of the essential physics (Haken 1983, p. 127). The dynamical variable is the number of photons $n(t)$ in the laser field. Its rate of change is given by

$$\begin{aligned}\dot{n} &= \text{gain} - \text{loss} \\ &= GnN - kn.\end{aligned}$$

The gain term comes from the process of *stimulated emission,* in which photons stimulate excited atoms to emit additional photons. Because this process occurs via random encounters between photons and excited atoms, it occurs at a rate proportional to n and to the number of excited atoms, denoted by $N(t)$. The parameter $G > 0$ is known as the gain coefficient. The loss term models the escape of photons through the endfaces of the laser. The parameter $k > 0$ is a rate constant; its reciprocal $\tau = 1/k$ represents the typical lifetime of a photon in the laser.

Now comes the key physical idea: after an excited atom emits a photon, it drops down to a lower energy level and is no longer excited. Thus N decreases by the emission of photons. To capture this effect, we need to write an equation relating N to n. Suppose that in the absence of laser action, the pump keeps the number of excited atoms fixed at N_0. Then the *actual* number of excited atoms will be reduced by the laser process. Specifically, we assume

$$N(t) = N_0 - \alpha n,$$

where $\alpha > 0$ is the rate at which atoms drop back to their ground states. Then

$$\begin{aligned}\dot{n} &= Gn(N_0 - \alpha n) - kn \\ &= (GN_0 - k)n - (\alpha G)n^2 .\end{aligned}$$

We're finally on familiar ground—this is a first-order system for $n(t)$. Figure 3.3.2 shows the corresponding vector field for different values of the pump strength N_0. Note that only positive values of n are physically meaningful.

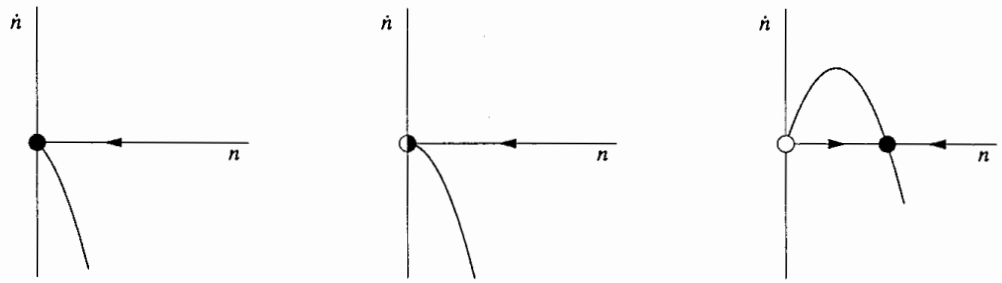


Figure 3.3.2

When $N_0 < k/G$, the fixed point at $n^* = 0$ is stable. This means that there is no stimulated emission and the laser acts like a lamp. As the pump strength N_0 is increased, the system undergoes a transcritical bifurcation when $N_0 = k/G$. For $N_0 > k/G$, the origin loses stability and a stable fixed point appears at $n^* = (GN_0 - k)/\alpha G > 0$, corresponding to spontaneous laser action. Thus $N_0 = k/G$ can be interpreted as the ***laser threshold*** in this model. Figure 3.3.3 summarizes our results.

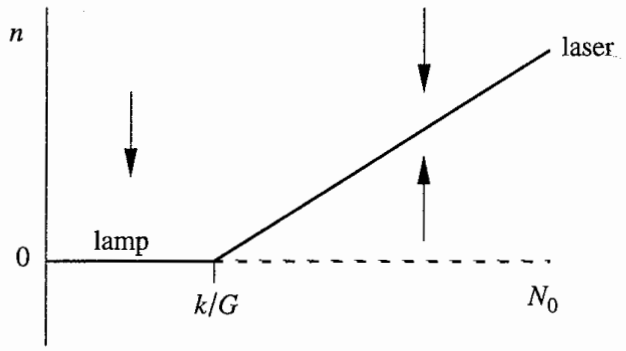


Figure 3.3.3

Although this model correctly predicts the existence of a threshold, it ignores the dynamics of the excited atoms, the existence of spontaneous emission, and several other complications. See Exercises 3.3.1 and 3.3.2 for improved models.

3.4 Pitchfork Bifurcation

We turn now to a third kind of bifurcation, the so-called pitchfork bifurcation. This bifurcation is common in physical problems that have a ***symmetry***. For example, many problems have a spatial symmetry between left and right. In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling example of Figure 3.0.1, the beam is stable in the vertical position if the load is small. In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to *either* the left or the right. The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

There are two very different types of pitchfork bifurcation. The simpler type is called *supercritical*, and will be discussed first.

Supercritical Pitchfork Bifurcation

The normal form of the supercritical pitchfork bifurcation is

$$\dot{x} = rx - x^3. \qquad (1)$$

Note that this equation is ***invariant*** under the change of variables $x \to -x$. That is, if we replace x by $-x$ and then cancel the resulting minus signs on both sides of the equation, we get (1) back again. This invariance is the mathematical expression of the left–right symmetry mentioned earlier. (More technically, one says that the vector field is *equivariant*, but we'll use the more familiar language.)

Figure 3.4.1 shows the vector field for different values of r.

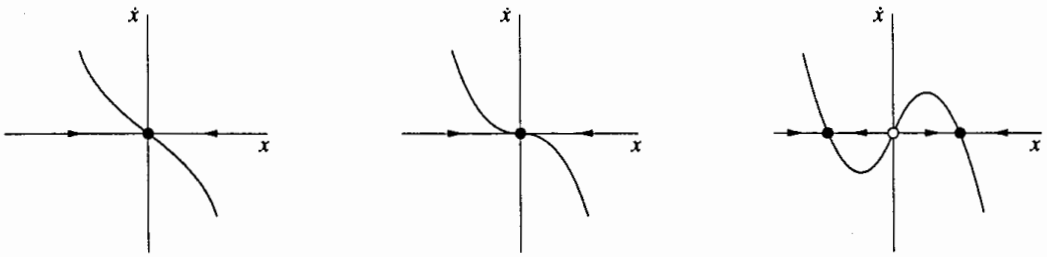


Figure 3.4.1

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast—instead the decay is a much slower algebraic function of time (recall Exercise 2.4.9). This lethargic decay is called ***critical slowing down*** in the physics literature. Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.

The reason for the term "pitchfork" becomes clear when we plot the bifurcation diagram (Figure 3.4.2). Actually, pitchfork trifurcation might be a better word!

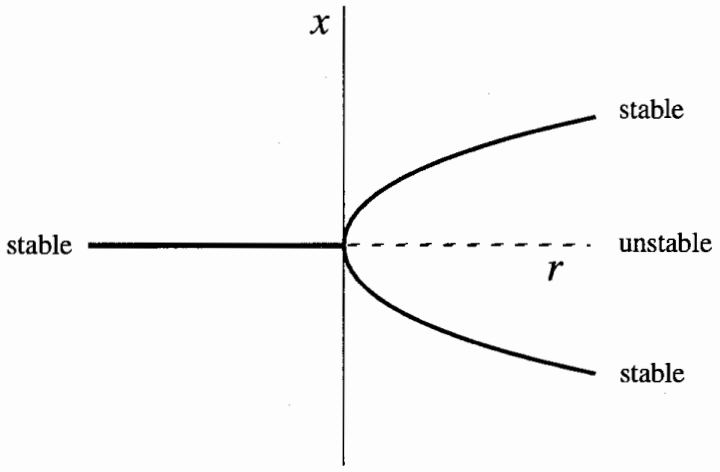


Figure 3.4.2

EXAMPLE 3.4.1:

Equations similar to $\dot{x} = -x + \beta \tanh x$ arise in statistical mechanical models of magnets and neural networks (see Exercise 3.6.7 and Palmer 1989). Show that this equation undergoes a supercritical pitchfork bifurcation as β is varied. Then give a *numerically accurate* plot of the fixed points for each β.

Solution: We use the strategy of Example 3.1.2 to find the fixed points. The graphs of $y = x$ and $y = \beta \tanh x$ are shown in Figure 3.4.3; their intersections correspond to fixed points. The key thing to realize is that as β increases, the tanh curve becomes steeper at the origin (its slope there is β). Hence for $\beta < 1$ the origin is the only fixed point. A pitchfork bifurcation occurs at $\beta = 1$, $x^* = 0$, when the tanh curve develops a slope of 1 at the origin. Finally, when $\beta > 1$, two new stable fixed points appear, and the origin becomes unstable.

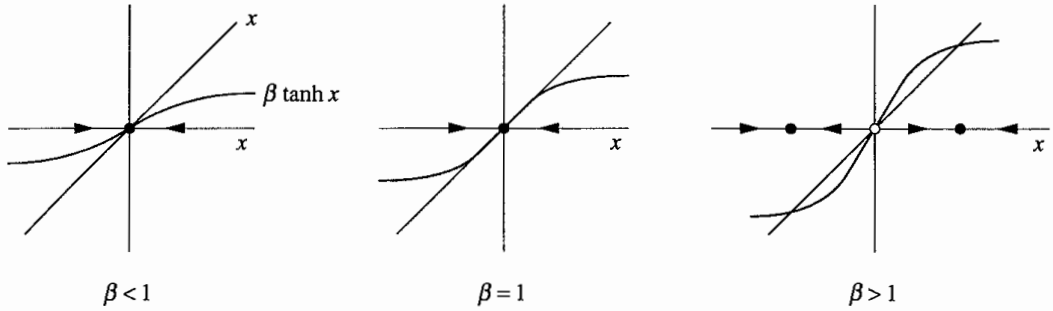


Figure 3.4.3

Now we want to compute the fixed points x^* for each β. Of course, one fixed point always occurs at $x^* = 0$; we are looking for the other, nontrivial fixed points.

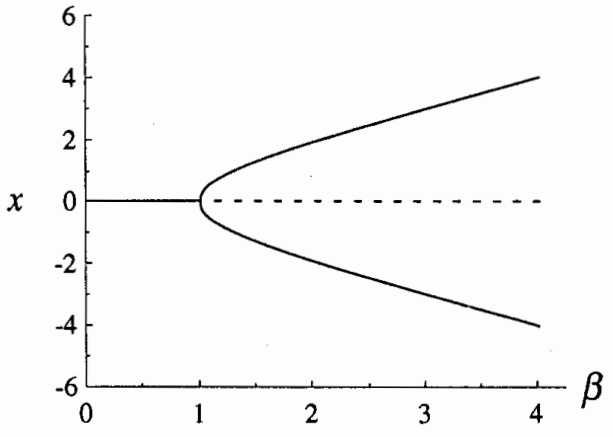


Figure 3.4.4

One approach is to solve the equation $x^* = \beta \tanh x^*$ numerically, using the Newton–Raphson method or some other root-finding scheme. (See Press et al. (1986) for a friendly and informative discussion of numerical methods.)

But there's an easier way, which comes from changing our point of view. Instead of studying the dependence of x^* on β, we think of x^* as the *independent* variable, and then compute $\beta = x^*/\tanh x^*$. This gives us a table of pairs (x^*, β). For each pair, we plot β horizontally and x^* vertically. This yields the bifurcation diagram (Figure 3.4.4).

The shortcut used here exploits the fact that $f(x,\beta) = -x + \beta \tanh x$ depends more simply on β than on x. This is frequently the case in bifurcation problems—the dependence on the control parameter is usually simpler than the dependence on x. ■

EXAMPLE 3.4.2:

Plot the potential $V(x)$ for the system $\dot{x} = rx - x^3$, for the cases $r < 0$, $r = 0$, and $r > 0$.

Solution: Recall from Section 2.7 that the potential for $\dot{x} = f(x)$ is defined by $f(x) = -dV/dx$. Hence we need to solve $-dV/dx = rx - x^3$. Integration yields $V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$, where we neglect the arbitrary constant of integration. The corresponding graphs are shown in Figure 3.4.5.

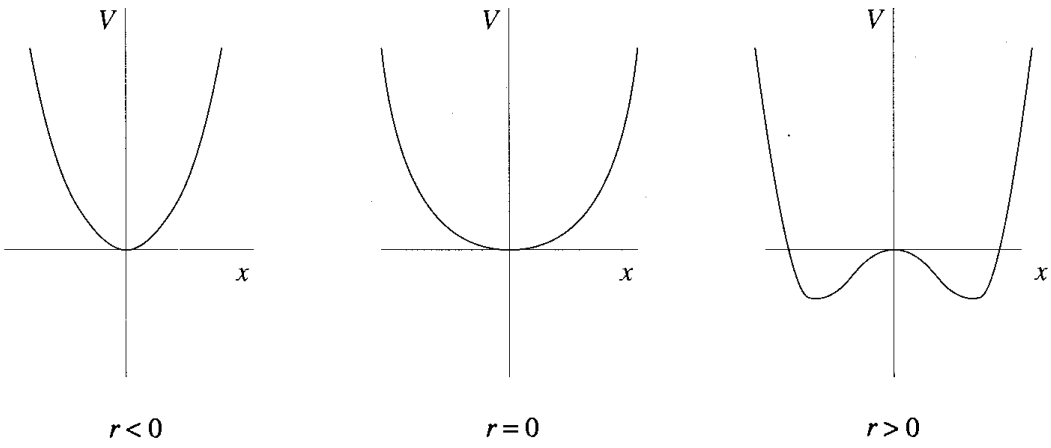


Figure 3.4.5

When $r < 0$, there is a quadratic minimum at the origin. At the bifurcation value $r = 0$, the minimum becomes a much flatter quartic. For $r > 0$, a local *maximum* appears at the origin, and a symmetric pair of minima occur to either side of it. ■

Subcritical Pitchfork Bifurcation

In the supercritical case $\dot{x} = rx - x^3$ discussed above, the cubic term is *stabilizing*: it acts as a restoring force that pulls $x(t)$ back toward $x = 0$. If instead the cubic term were *destabilizing*, as in

$$\dot{x} = rx + x^3, \qquad (2)$$

then we'd have a ***subcritical*** pitchfork bifurcation. Figure 3.4.6 shows the bifurcation diagram.

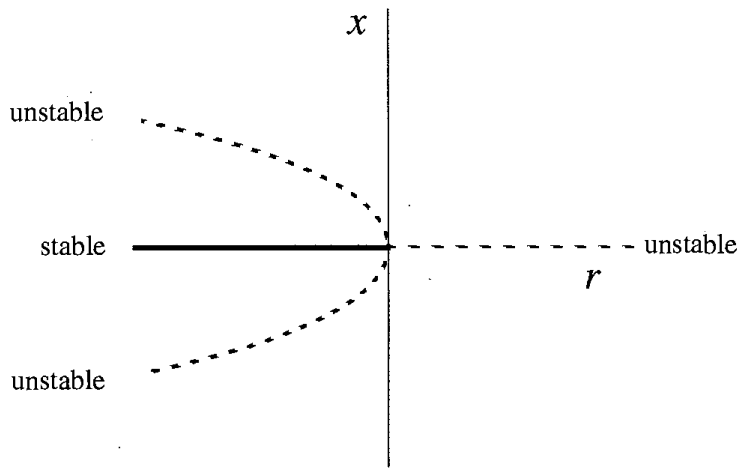


Figure 3.4.6

Compared to Figure 3.4.2, the pitchfork is inverted. The nonzero fixed points $x^* = \pm\sqrt{-r}$ are *unstable,* and exist only *below* the bifurcation ($r < 0$), which motivates the term "subcritical." More importantly, the origin is stable for $r < 0$ and unstable for $r > 0$, as in the supercritical case, but now the instability for $r > 0$ is not opposed by the cubic term—in fact the cubic term lends a helping hand in driving the trajectories out to infinity! This effect leads to *blow-up*: one can show that $x(t) \to \pm\infty$ in finite time, starting from any initial condition $x_0 \neq 0$ (Exercise 2.5.3).

In real physical systems, such an explosive instability is usually opposed by the stabilizing influence of higher-order terms. Assuming that the system is still symmetric under $x \to -x$, the first stabilizing term must be x^5. Thus the canonical example of a system with a subcritical pitchfork bifurcation is

$$\dot{x} = rx + x^3 - x^5. \tag{3}$$

There's no loss in generality in assuming that the coefficients of x^3 and x^5 are 1 (Exercise 3.5.8).

The detailed analysis of (3) is left to you (Exercises 3.4.14 and 3.4.15). But we will summarize the main results here. Figure 3.4.7 shows the bifurcation diagram for (3).

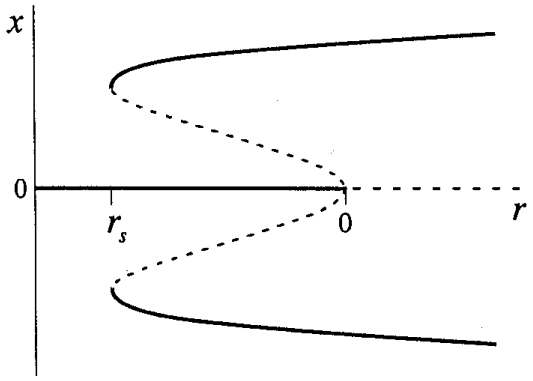


Figure 3.4.7

For small x, the picture looks just like Figure 3.4.6: the origin is locally stable for $r < 0$, and two backward-bending branches of unstable fixed points bifurcate from the origin when $r = 0$. The new feature, due to the x^5 term, is that the unstable branches turn around and become stable at $r = r_s$, where $r_s < 0$. These stable ***large-amplitude*** branches exist for all $r > r_s$.

There are several things to note about Figure 3.4.7:

1. In the range $r_s < r < 0$, two qualitatively different stable states coexist, namely the origin and the large-amplitude fixed points. The initial condition x_0 determines which fixed point is approached as $t \to \infty$. One consequence is that the origin is stable to small perturbations, but not to large ones—in this sense the origin is *locally* stable, but not *globally* stable.
2. The existence of different stable states allows for the possibility of ***jumps*** and ***hysteresis*** as r is varied. Suppose we start the system in the state $x^* = 0$, and then slowly increase the parameter r (indicated by an arrow along the r-axis of Figure 3.4.8).

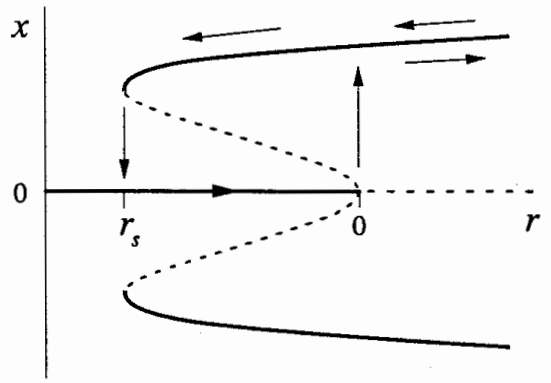


Figure 3.4.8

Then the state remains at the origin until $r = 0$, when the origin loses stability. Now the slightest nudge will cause the state to *jump* to one of the large-amplitude branches. With further increases of r, the state moves out along the large-amplitude branch. If r is now decreased, the state remains on the large-amplitude branch, even when r is decreased below 0! We have to lower r even further (down past r_s) to get the state to jump back to the origin. This lack of reversibility as a parameter is varied is called *hysteresis.*

3. The bifurcation at r_s is a saddle-node bifurcation, in which stable and unstable fixed points are born "out the clear blue sky" as r is increased (see Section 3.1).

Terminology

As usual in bifurcation theory, there are several other names for the bifurcations discussed here. The supercritical pitchfork is sometimes called a forward bifurcation, and is closely related to a continuous or second-order phase transition in sta-

tistical mechanics. The subcritical bifurcation is sometimes called an inverted or backward bifurcation, and is related to discontinuous or first-order phase transitions. In the engineering literature, the supercritical bifurcation is sometimes called soft or safe, because the nonzero fixed points are born at small amplitude; in contrast, the subcritical bifurcation is hard or dangerous, because of the jump from zero to large amplitude.

3.5 Overdamped Bead on a Rotating Hoop

In this section we analyze a classic problem from first-year physics, the bead on a rotating hoop. This problem provides an example of a bifurcation in a mechanical system. It also illustrates the subtleties involved in replacing Newton's law, which is a second-order equation, by a simpler first-order equation.

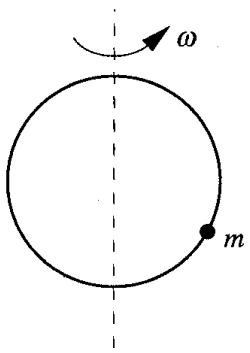


Figure 3.5.1

The mechanical system is shown in Figure 3.5.1. A bead of mass m slides along a wire hoop of radius r. The hoop is constrained to rotate at a constant angular velocity ω about its vertical axis. The problem is to analyze the motion of the bead, given that it is acted on by both gravitational and centrifugal forces. This is the usual statement of the problem, but now we want to add a new twist: suppose that there's also a frictional force on the bead that opposes its motion. To be specific, imagine that the whole system is immersed in a vat of molasses or some other very viscous fluid, and that the friction is due to viscous damping.

Let ϕ be the angle between the bead and the downward vertical direction. By convention, we restrict ϕ to the range $-\pi < \phi \leq \pi$, so there's only one angle for each point on the hoop. Also, let $\rho = r\sin\phi$ denote the distance of the bead from the vertical axis. Then the coordinates are as shown in Figure 3.5.2.

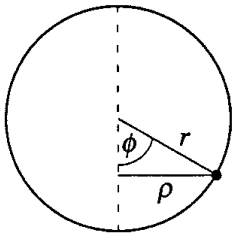


Figure 3.5.2

Now we write Newton's law for the bead. There's a downward gravitational force mg, a sideways centrifugal force $m\rho\omega^2$, and a tangential damping force $b\dot{\phi}$. (The constants g and b are taken to be positive; negative signs will be added later as needed.) The hoop is assumed to be rigid, so we only have to resolve the forces along the tangential direction, as shown in Figure 3.5.3. After substituting $\rho = r\sin\phi$ in the centrifugal term, and recalling that the tangential acceleration is $r\ddot{\phi}$, we obtain the governing equation

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi. \qquad (1)$$

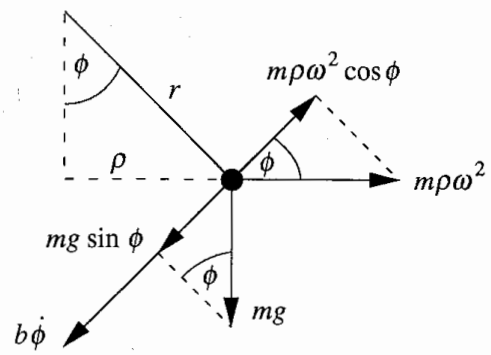


Figure 3.5.3

This is a *second-order* differential equation, since the second derivative $\ddot{\phi}$ is the highest one that appears. We are not yet equipped to analyze second-order equations, so we would like to find some conditions under which we can safely neglect the $mr\ddot{\phi}$ term. Then (1) reduces to a first-order equation, and we can apply our machinery to it.

Of course, this is a dicey business: we can't just neglect terms because we feel like it! But we will for now, and then at the end of this section we'll try to find a regime where our approximation is valid.

Analysis of the First-Order System

Our concern now is with the first-order system

$$b\dot{\phi} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi$$

$$= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \qquad (2)$$

The fixed points of (2) correspond to equilibrium positions for the bead. What's your intuition about where such equilibria can occur? We would expect the bead to remain at rest if placed at the top or the bottom of the hoop. Can other fixed points occur? And what about stability? Is the bottom always stable?

Equation (2) shows that there are always fixed points where $\sin\phi = 0$, namely $\phi^* = 0$ (the bottom of the hoop) and $\phi^* = \pi$ (the top). The more interesting result is that there are two *additional* fixed points if

$$\frac{r\omega^2}{g} > 1,$$

that is, *if the hoop is spinning fast enough.* These fixed points satisfy $\phi^* = \pm\cos^{-1}\left(g/r\omega^2\right)$. To visualize them, we introduce a parameter

$$\gamma = \frac{r\omega^2}{g}$$

and solve $\cos\phi^* = 1/\gamma$ graphically. We plot $\cos\phi$ vs. ϕ, and look for intersections with the constant function $1/\gamma$, shown as a horizontal line in Figure 3.5.4. For $\gamma < 1$ there are no intersections, whereas for $\gamma > 1$ there is a symmetrical pair of in-

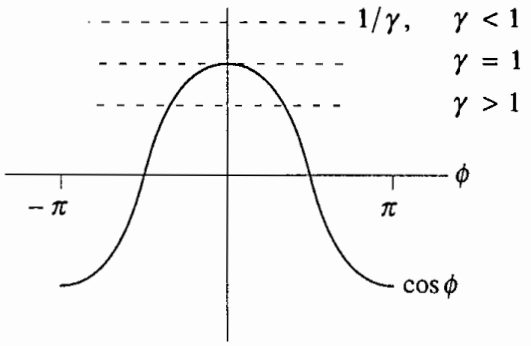


Figure 3.5.4

tersections to either side of $\phi^* = 0$. As $\gamma \to \infty$, these intersections approach $\pm\pi/2$. Figure 3.5.5 plots the fixed points on the hoop for the cases $\gamma < 1$ and $\gamma > 1$.

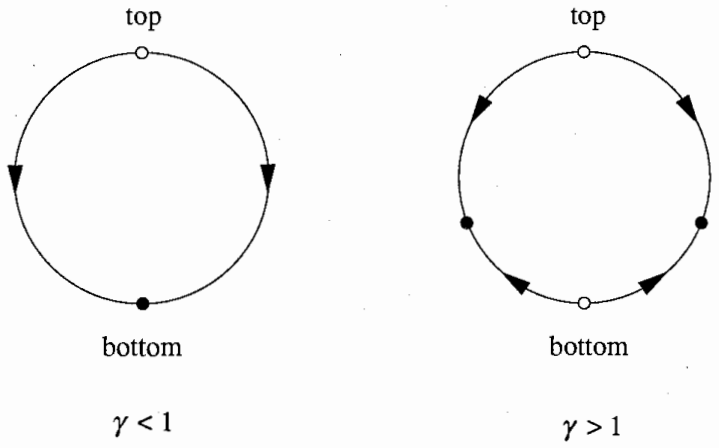


Figure 3.5.5

To summarize our results so far, let's plot *all* the fixed points as a function of the parameter γ (Figure 3.5.6). As usual, solid lines denote stable fixed points and broken lines denote unstable fixed points.

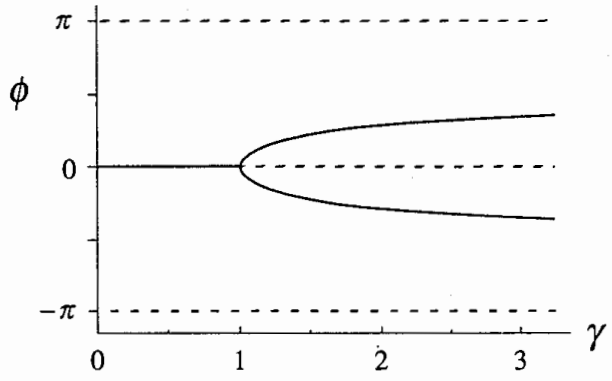


Figure 3.5.6

We now see that a ***supercritical pitchfork bifurcation*** occurs at $\gamma = 1$. It's left to you to check the stability of the fixed points, using linear stability analysis or graphical methods (Exercise 3.5.2).

Here's the physical interpretation of the results: When $\gamma < 1$, the hoop is rotating slowly and the centrifugal force is too weak to balance the force of gravity. Thus the bead slides down to the bottom and stays there. But if $\gamma > 1$, the hoop is spinning fast enough that the bottom becomes unstable. Since the centrifugal force *grows* as the bead moves farther from the bottom, any slight displacement of the bead will be *amplified.* The bead is therefore pushed up the hoop until gravity balances the centrifugal force; this balance occurs at $\phi^* = \pm\cos^{-1}\left(g/r\omega^2\right)$. Which of these two fixed points is actually selected depends on the initial disturbance. Even though the two fixed points are entirely symmetrical, an asymmetry in the initial conditions will lead to one of them being chosen—physicists sometimes refer to these as ***symmetry-broken*** solutions. In other words, the solution has less symmetry than the governing equation.

What *is* the symmetry of the governing equation? Clearly the left and right halves of the hoop are physically equivalent—this is reflected by the invariance of (1) and (2) under the change of variables $\phi \to -\phi$. As we mentioned in Section 3.4, pitchfork bifurcations are to be expected in situations where such a symmetry exists.

Dimensional Analysis and Scaling

Now we need to address the question: When is it valid to neglect the inertia term $mr\ddot{\phi}$ in (1)? At first sight the limit $m \to 0$ looks promising, but then we notice that we're throwing out the baby with the bathwater: the centrifugal and gravitational terms vanish in this limit too! So we have to be more careful.

In problems like this, it is helpful to express the equation in ***dimensionless*** form (at present, all the terms in (1) have the dimensions of force.) The advantage of a dimensionless formulation is that we know how to define *small*—it means "much less than 1." Furthermore, nondimensionalizing the equation reduces the number of parameters by lumping them together into ***dimensionless groups.*** This reduction always simplifies the analysis. For an excellent introduction to dimensional analysis, see Lin and Segel (1988).

There are often several ways to nondimensionalize an equation, and the best choice might not be clear at first. Therefore we proceed in a flexible fashion. We define a dimensionless time τ by

$$\tau = \frac{t}{T}$$

where T is a ***characteristic time scale*** to be chosen later. When T is chosen correctly, the new derivatives $d\phi/d\tau$ and $d^2\phi/d\tau^2$ should be $O(1)$, i.e., of order

unity. To express these new derivatives in terms of the old ones, we use the chain rule:

$$\dot{\phi} \equiv \frac{d\phi}{dt} = \frac{d\phi}{d\tau}\frac{d\tau}{dt} = \frac{1}{T}\frac{d\phi}{d\tau}$$

and similarly

$$\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}.$$

(The easy way to remember these formulas is to formally substitute $T\tau$ for t.) Hence (1) becomes

$$\frac{mr}{T^2}\frac{d^2\phi}{d\tau^2} = -\frac{b}{T}\frac{d\phi}{d\tau} - mg\sin\phi + mr\omega^2\sin\phi\cos\phi.$$

Now since this equation is a balance of forces, we nondimensionalize it by dividing by a force mg. This yields the dimensionless equation

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi. \qquad (3)$$

Each of the terms in parentheses is a dimensionless group. We recognize the group $r\omega^2/g$ in the last term—that's our old friend γ from earlier in the section.

We are interested in the regime where the left-hand side of (3) is negligible compared to all the other terms, and where all the terms on the right-hand side are of comparable size. Since the derivatives are $O(1)$ by assumption, and $\sin\phi \approx O(1)$, we see that we need

$$\frac{b}{mgT} \approx O(1), \text{ and } \frac{r}{gT^2} << 1.$$

The first of these requirements sets the time scale T: a natural choice is

$$T = \frac{b}{mg}.$$

Then the condition $r/gT^2 << 1$ becomes

$$\frac{r}{g}\left(\frac{mg}{b}\right)^2 << 1, \qquad (4)$$

or equivalently,

$$b^2 >> m^2 gr.$$

This can be interpreted as saying that the *damping is very strong*, or that the mass is very small, now in a precise sense.

The condition (4) motivates us to introduce a dimensionless group

$$\varepsilon = \frac{m^2 g r}{b^2} . \tag{5}$$

Then (3) becomes

$$\varepsilon \frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma \sin\phi \cos\phi . \tag{6}$$

As advertised, the dimensionless Equation (6) is simpler than (1): the five parameters m, g, r, ω, and b have been replaced by two dimensionless groups γ and ε.

In summary, our dimensional analysis suggests that in the ***overdamped*** limit $\varepsilon \to 0$, (6) should be well approximated by the first-order system

$$\frac{d\phi}{d\tau} = f(\phi) \tag{7}$$

where

$$\begin{aligned} f(\phi) &= -\sin\phi + \gamma \sin\phi \cos\phi \\ &= \sin\phi(\gamma \cos\phi - 1). \end{aligned}$$

A Paradox

Unfortunately, *there is something fundamentally wrong with our idea of replacing a second-order equation by a first-order equation.* The trouble is that a second-order equation requires *two* initial conditions, whereas a first-order equation has only *one*. In our case, the bead's motion is determined by its initial position and velocity. These two quantities can be chosen completely independent of each other. But that's not true for the first-order system: given the initial position, the initial velocity is dictated by the equation $d\phi/d\tau = f(\phi)$. Thus the solution to the first-order system will not, in general, be able to satisfy *both* initial conditions.

We seem to have run into a paradox. Is (7) valid in the overdamped limit or not? If it is valid, how can we satisfy the two arbitrary initial conditions demanded by (6)?

The resolution of the paradox requires us to analyze the second-order system (6). We haven't dealt with second-order systems before—that's the subject of Chapter 5. But read on if you're curious; some simple ideas are all we need to finish the problem.

Phase Plane Analysis

Throughout Chapters 2 and 3, we have exploited the idea that a first-order sys-

tem $\dot{x} = f(x)$ can be regarded as a vector field on a line. By analogy, the *second-order* system (6) can be regarded as a vector field on a *plane*, the so-called ***phase plane.***

The plane is spanned by two axes, one for the angle ϕ and one for the angular velocity $d\phi/d\tau$. To simplify the notation, let

$$\Omega = \phi' \equiv d\phi/d\tau$$

where prime denotes differentiation with respect to τ. Then an initial condition for (6) corresponds to a point (ϕ_0, Ω_0) in the phase plane (Figure 3.5.7). As time evolves, the phase point $(\phi(t), \Omega(t))$ moves around in the phase plane along a ***trajectory*** determined by the solution to (6).

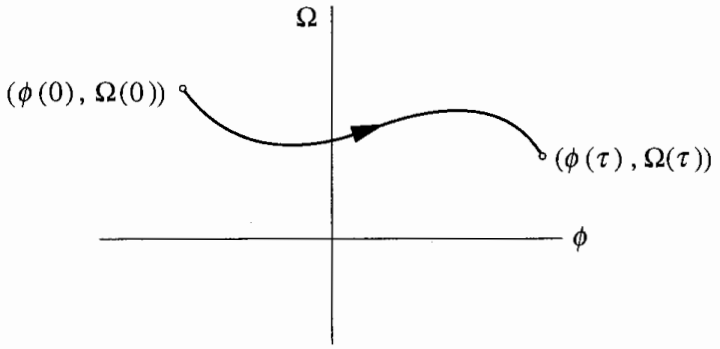


Figure 3.5.7

Our goal now is to see what those trajectories actually look like. As before, the key idea is that *the differential equation can be interpreted as a vector field on the phase space.* To convert (6) into a vector field, we first rewrite it as

$$\varepsilon\Omega' = f(\phi) - \Omega.$$

Along with the definition $\phi' = \Omega$, this yields the ***vector field***

$$\phi' = \Omega \tag{8a}$$

$$\Omega' = \frac{1}{\varepsilon}\big(f(\phi) - \Omega\big). \tag{8b}$$

We interpret the vector (ϕ', Ω') at the point (ϕ, Ω) as the local velocity of a phase fluid flowing steadily on the plane. Note that the velocity vector now has two components, one in the ϕ-direction and one in the Ω-direction. To visualize the trajectories, we just imagine how the phase point would move as it is carried along by the phase fluid.

In general, the pattern of trajectories would be difficult to picture, but the pre-

sent case is simple because we are only interested in the limit $\varepsilon \to 0$. In this limit, *all trajectories slam straight up or down onto the curve C defined by $f(\phi) = \Omega$, and then slowly ooze along this curve until they reach a fixed point* (Figure 3.5.8).

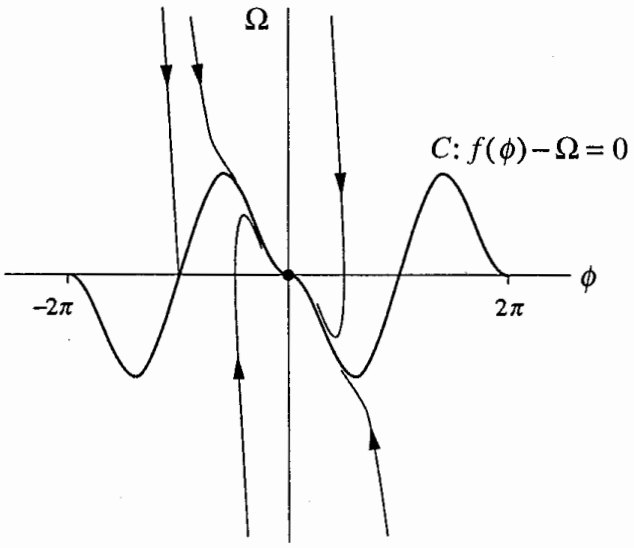


Figure 3.5.8

To arrive at this striking conclusion, let's do an order-of-magnitude calculation. Suppose that the phase point lies off the curve C. For instance, suppose (ϕ, Ω) lies an $O(1)$ distance below the curve C, i.e., $\Omega < f(\phi)$ and $f(\phi) - \Omega \approx O(1)$. Then (8b) shows that Ω' is enormously positive: $\Omega' \approx O(1/\varepsilon) >> 1$. Thus the phase point zaps like lightning up to the region where $f(\phi) - \Omega \approx O(\varepsilon)$. In the limit $\varepsilon \to 0$, this region is indistinguishable from C. Once the phase point is on C, it evolves according to $\Omega \approx f(\phi)$; that is, it approximately satisfies the first-order equation $\phi' = f(\phi)$.

Our conclusion is that a typical trajectory is made of two parts: a rapid initial ***transient***, during which the phase point zaps onto the curve where $\phi' = f(\phi)$, followed by a much slower drift along this curve.

Now we see how the paradox is resolved: The second-order system (6) *does* behave like the first-order system (7), but only after a rapid initial transient. During this transient, it is *not* correct to neglect the term $\varepsilon d^2\phi/d\tau^2$. The problem with our earlier approach is that we used only a single time scale $T = b/mg$; this time scale is characteristic of the slow drift process, but not of the rapid transient (Exercise 3.5.5).

A Singular Limit

The difficulty we have encountered here occurs throughout science and engineering. In some limit of interest (here, the limit of strong damping), the term con-

taining the highest order derivative drops out of the governing equation. Then the initial conditions or boundary conditions can't be satisfied. Such a limit is often called *singular*. For example, in fluid mechanics, the limit of high Reynolds number is a singular limit; it accounts for the presence of extremely thin "boundary layers" in the flow over airplane wings. In our problem, the rapid transient played the role of a boundary layer—it is a thin layer of *time* that occurs near the boundary $t = 0$.

The branch of mathematics that deals with singular limits is called *singular perturbation theory*. See Jordan and Smith (1987) or Lin and Segel (1988) for an introduction. Another problem with a singular limit will be discussed briefly in Section 7.5.

3.6 Imperfect Bifurcations and Catastrophes

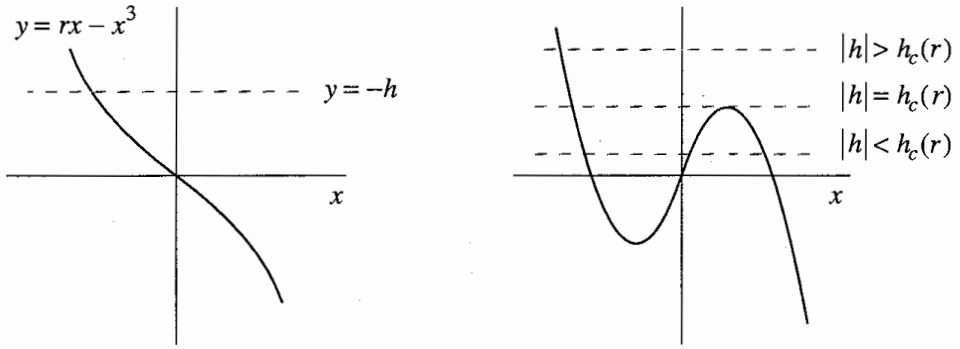
As we mentioned earlier, pitchfork bifurcations are common in problems that have a symmetry. For example, in the problem of the bead on a rotating hoop (Section 3.5), there was a perfect symmetry between the left and right sides of the hoop. But in many real-world circumstances, the symmetry is only approximate—an imperfection leads to a slight difference between left and right. We now want to see what happens when such imperfections are present.

For example, consider the system

$$\dot{x} = h + rx - x^3. \qquad (1)$$

If $h = 0$, we have the normal form for a supercritical pitchfork bifurcation, and there's a perfect symmetry between x and $-x$. But this symmetry is broken when $h \neq 0$; for this reason we refer to h as an ***imperfection parameter***.

Equation (1) is a bit harder to analyze than other bifurcation problems we've considered previously, because we have *two* independent parameters to worry about (h and r). To keep things straight, we'll think of r as fixed, and then examine the effects of varying h. The first step is to analyze the fixed points of (1). These can be found explicitly, but we'd have to invoke the messy formula for the roots of a cubic equation. It's clearer to use a graphical approach, as in Example 3.1.2. We plot the graphs of $y = rx - x^3$ and $y = -h$ on the same axes, and look for intersections (Figure 3.6.1). These intersections occur at the fixed points of (1). When $r \leq 0$, the cubic is monotonically decreasing, and so it intersects the horizontal line $y = -h$ in exactly one point (Figure 3.6.1a). The more interesting case is $r > 0$; then one, two, or three intersections are possible, depending on the value of h (Figure 3.6.1b).



(a) $r \le 0$ (b) $r > 0$

Figure 3.6.1

The critical case occurs when the horizontal line is just *tangent* to either the local minimum or maximum of the cubic; then we have a *saddle-node bifurcation.* To find the values of h at which this bifurcation occurs, note that the cubic has a local maximum when $\frac{d}{dx}(rx - x^3) = r - 3x^2 = 0$. Hence

$$x_{\max} = \sqrt{\frac{r}{3}},$$

and the value of the cubic at the local maximum is

$$rx_{\max} - (x_{\max})^3 = \frac{2r}{3}\sqrt{\frac{r}{3}}.$$

Similarly, the value at the minimum is the negative of this quantity. Hence saddle-node bifurcations occur when $h = \pm h_c(r)$, where

$$h_c(r) = \frac{2r}{3}\sqrt{\frac{r}{3}}.$$

Equation (1) has three fixed points for $|h| < h_c(r)$ and one fixed point for $|h| > h_c(r)$.

To summarize the results so far, we plot the ***bifurcation curves*** $h = \pm h_c(r)$ in the (r,h) plane (Figure 3.6.2). Note that the two bifurcation curves meet tangentially at $(r,h) = (0,0)$; such a point is called a ***cusp point***. We also label the regions that correspond to different numbers of fixed points. Saddle-node bifurcations occur all along the boundary of the regions, except at the cusp point, where we have a *codimension-2 bifurcation.* (This fancy terminology essentially means that we have had to tune *two* parameters, h and r, to achieve this type of bifurcation. Until now, all our bifurcations could be achieved by tuning a single parameter, and were therefore *codimension-1* bifurcations.)

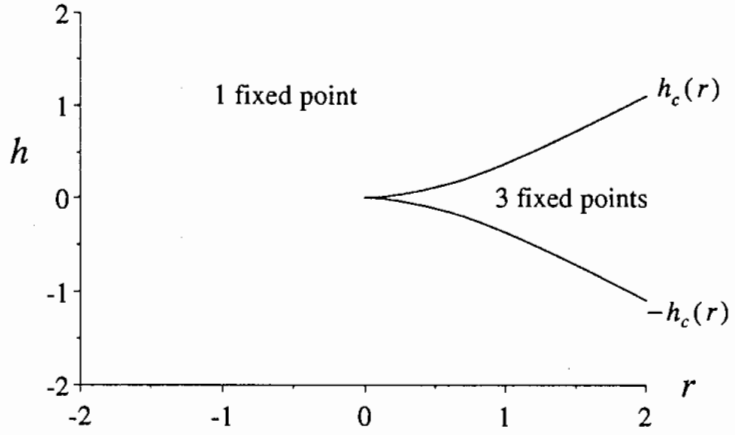
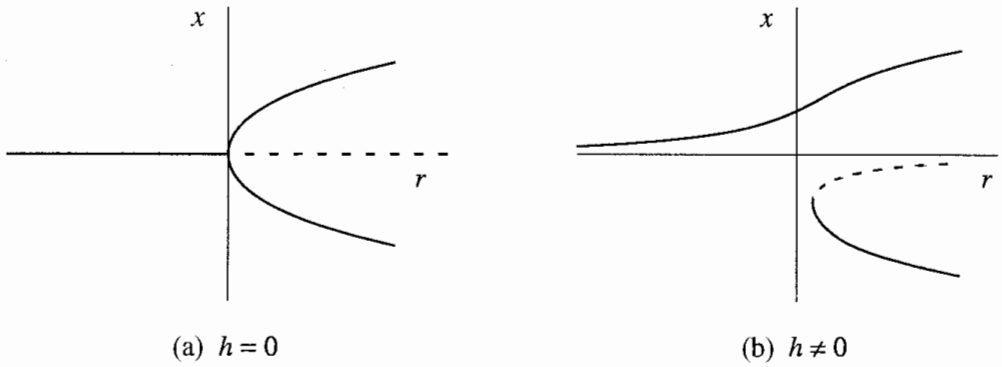


Figure 3.6.2

Pictures like Figure 3.6.2 will prove very useful in our future work. We will refer to such pictures as ***stability diagrams***. They show the different types of behavior that occur as we move around in ***parameter space*** (here, the (r,h) plane).

Now let's present our results in a more familiar way by showing the bifurcation diagram of x^* vs. r, for fixed h (Figure 3.6.3).

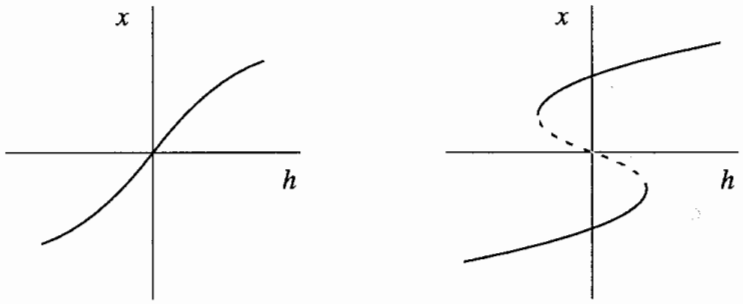


(a) $h = 0$ (b) $h \neq 0$

Figure 3.6.3

When $h = 0$ we have the usual pitchfork diagram (Figure 3.6.3a) but when $h \neq 0$, the pitchfork disconnects into two pieces (Figure 3.6.3b). The upper piece consists entirely of stable fixed points, whereas the lower piece has both stable and unstable branches. As we increase r from negative values, there's no longer a sharp transition at $r = 0$; the fixed point simply glides smoothly along the upper branch. Furthermore, the lower branch of stable points is not accessible unless we make a fairly large disturbance.

Alternatively, we could plot x^* vs. h, for fixed r (Figure 3.6.4).



(a) $r \le 0$ (b) $r > 0$

Figure 3.6.4

When $r \le 0$ there's one stable fixed point for each h (Figure 3.6.4a). However, when $r > 0$ there are three fixed points when $|h| < h_c(r)$, and one otherwise (Figure 3.6.4b). In the triple-valued region, the middle branch is unstable and the upper and lower branches are stable. Note that these graphs look like Figure 3.6.1 rotated by 90°.

There is one last way to plot the results, which may appeal to you if you like to picture things in three dimensions. This method of presentation contains all of the others as cross sections or projections. If we plot the fixed points x^* above the (r,h) plane, we get the ***cusp catastrophe*** surface shown in Figure 3.6.5. The surface folds over on itself in certain places. The projection of these folds onto the (r,h) plane yields the bifurcation curves shown in Figure 3.6.2. A cross section at fixed h yields Figure 3.6.3, and a cross section at fixed r yields Figure 3.6.4.

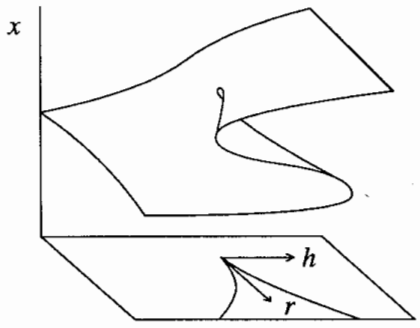


Figure 3.6.5

The term *catastrophe* is motivated by the fact that as parameters change, the state of the system can be carried over the edge of the upper surface, after which it drops discontinuously to the lower surface (Figure 3.6.6). This jump could be truly catastrophic for the equilibrium of a bridge or a building. We will see scientific examples of catastrophes in the context of insect outbreaks (Section 3.7) and in the following example from mechanics.

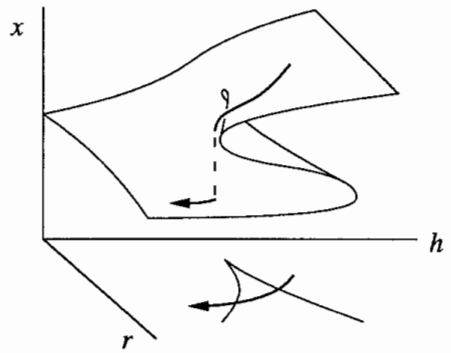


Figure 3.6.6

For more about catastrophe theory, see Zeeman (1977) or Poston and Stewart (1978). Incidentally, there was a violent controversy about this subject in the late

1970s. If you like watching fights, have a look at Zahler and Sussman (1977) and Kolata (1977).

Bead on a Tilted Wire

As a simple example of imperfect bifurcation and catastrophe, consider the following mechanical system (Figure 3.6.7).

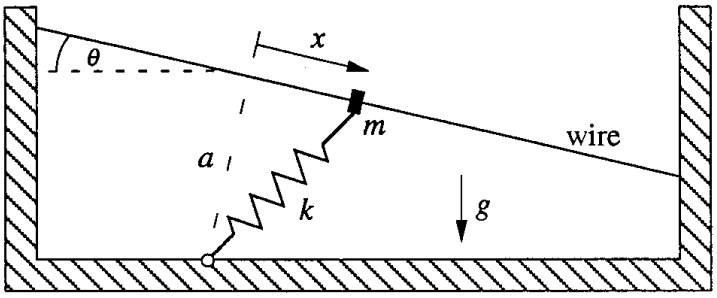


Figure 3.6.7

A bead of mass m is constrained to slide along a straight wire inclined at an angle θ with respect to the horizontal. The mass is attached to a spring of stiffness k and relaxed length L_0, and is also acted on by gravity. We choose coordinates along the wire so that $x = 0$ occurs at the point closest to the support point of the spring; let a be the distance between this support point and the wire.

In Exercises 3.5.4 and 3.6.5, you are asked to analyze the equilibrium positions of the bead. But first let's get some physical intuition. When the wire is horizontal ($\theta = 0$), there is perfect symmetry between the left and right sides of the wire, and $x = 0$ is always an equilibrium position. The stability of this equilibrium depends on the relative sizes of L_0 and a: if $L_0 < a$, the spring is in tension and so the equilibrium should be stable. But if $L_0 > a$, the spring is compressed and so we expect an *unstable* equilibrium at $x = 0$ and a pair of stable equilibria to either side of it. Exercise 3.5.4 deals with this simple case.

The problem becomes more interesting when we tilt the wire ($\theta \neq 0$). For small tilting, we expect that there are still three equilibria if $L_0 > a$. However if the tilt becomes too steep, perhaps you can see intuitively that the uphill equilibrium might suddenly disappear, causing the bead to jump catastrophically to the downhill equilibrium. You might even want to build this mechanical system and try it. Exercise 3.6.5 asks you to work through the mathematical details.

3.7 Insect Outbreak

For a biological example of bifurcation and catastrophe, we turn now to a model for the sudden outbreak of an insect called the spruce budworm. This insect is a se-

rious pest in eastern Canada, where it attacks the leaves of the balsam fir tree. When an outbreak occurs, the budworms can defoliate and kill most of the fir trees in the forest in about four years.

Ludwig et al. (1978) proposed and analyzed an elegant model of the interaction between budworms and the forest. They simplified the problem by exploiting a separation of time scales: the budworm population evolves on a *fast* time scale (they can increase their density fivefold in a year, so they have a characteristic time scale of months), whereas the trees grow and die on a *slow* time scale (they can completely replace their foliage in about 7–10 years, and their life span in the absence of budworms is 100–150 years.) Thus, as far as the budworm dynamics are concerned, the forest variables may be treated as constants. At the end of the analysis, we will allow the forest variables to drift very slowly—this drift ultimately triggers an outbreak.

Model

The proposed model for the budworm population dynamics is

$$\dot{N} = RN\left(1-\frac{N}{K}\right) - p(N).$$

In the absence of predators, the budworm population $N(t)$ is assumed to grow logistically with growth rate R and carrying capacity K. The carrying capacity depends on the amount of foliage left on the trees, and so it is a slowly drifting parameter; at this stage we treat it as fixed. The term $p(N)$ represents the death rate due to *predation*, chiefly by birds, and is assumed to have the shape shown in Figure 3.7.1. There is almost no predation when budworms are scarce; the birds seek food elsewhere. However, once the population exceeds a certain critical level $N = A$, the predation turns on sharply and then saturates (the birds are eating as fast as they can). Ludwig et al. (1978) assumed the specific form

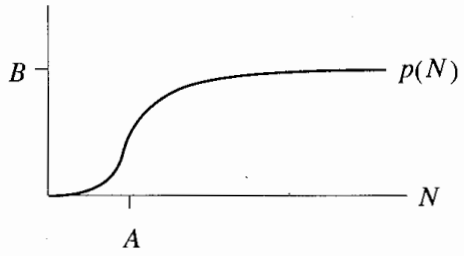


Figure 3.7.1

$$p(N) = \frac{BN^2}{A^2+N^2}$$

where $A, B > 0$. Thus the full model is

$$\dot{N} = RN\left(1-\frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}. \qquad (1)$$

We now have several questions to answer. What do we mean by an "outbreak" in the context of this model? The idea must be that, as parameters drift, the bud-

worm population suddenly jumps from a low to a high level. But what do we mean by "low" and "high," and are there solutions with this character? To answer these questions, it is convenient to recast the model into a dimensionless form, as in Section 3.5.

Dimensionless Formulation

The model (1) has four parameters: R, K, A, and B. As usual, there are various ways to nondimensionalize the system. For example, both A and K have the same dimension as N, and so either N/A or N/K could serve as a dimensionless population level. It often takes some trial and error to find the best choice. In this case, our heuristic will be to scale the equation so that all the dimensionless groups are pushed into the *logistic* part of the dynamics, with none in the *predation* part. This turns out to ease the graphical analysis of the fixed points.

To get rid of the parameters in the predation term, we divide (1) by B and then let

$$x = N/A,$$

which yields

$$\frac{A}{B}\frac{dx}{dt} = \frac{R}{B}Ax\left(1-\frac{Ax}{K}\right)-\frac{x^2}{1+x^2}. \qquad (2)$$

Equation (2) suggests that we should introduce a dimensionless time τ and dimensionless groups r and k, as follows:

$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}.$$

Then (2) becomes,

$$\frac{dx}{d\tau} = rx\left(1-\frac{x}{k}\right)-\frac{x^2}{1+x^2}, \qquad (3)$$

which is our final dimensionless form. Here r and k are the dimensionless growth rate and carrying capacity, respectively.

Analysis of Fixed Points

Equation (3) has a fixed point at $x^* = 0$; it is *always unstable* (Exercise 3.7.1). The intuitive explanation is that the predation is extremely weak for small x, and so the budworm population grows exponentially for x near zero.

The other fixed points of (3) are given by the solutions of

$$r\left(1-\frac{x}{k}\right) = \frac{x}{1+x^2}. \qquad (4)$$

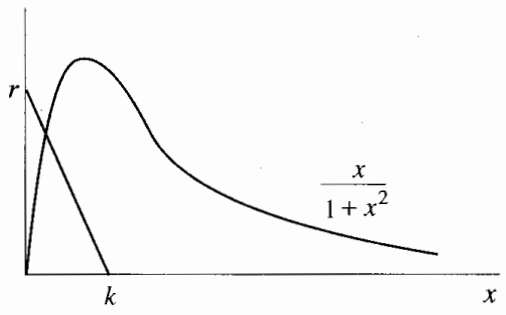


Figure 3.7.2

This equation is easy to analyze graphically—we simply graph the right- and left-hand sides of (4), and look for intersections (Figure 3.7.2). The left-hand side of (4) represents a straight line with x-intercept equal to k and a y-intercept equal to r, and the right-hand side represents a curve that is *independent of the parameters*! Hence, as we vary the parameters r and k, the line moves but the curve doesn't—this convenient property is what motivated our choice of nondimensionalization.

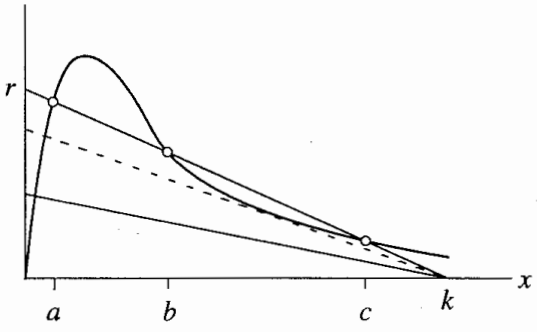


Figure 3.7.3

Figure 3.7.2 shows that if k is sufficiently small, there is exactly one intersection for any $r > 0$. However, for large k, we can have one, two, or three intersections, depending on the value of r (Figure 3.7.3). Let's suppose that there are three intersections a, b, and c. As we decrease r with k fixed, the line rotates counterclockwise about k. Then the fixed points b and c approach each other and eventually coalesce in a *saddle-node bifurcation* when the line intersects the curve *tangentially* (dashed line in Figure 3.7.3). After the bifurcation, the only remaining fixed point is a (in addition to $x^* = 0$, of course). Similarly, a and b can collide and annihilate as r is *increased.*

To determine the stability of the fixed points, we recall that $x^* = 0$ is unstable, and also observe that the stability type must alternate as we move along the x-axis. Hence a is stable, b is unstable, and c is stable. Thus, for r and k in the range corresponding to three positive fixed points, the vector field is qualitatively like that shown in Figure 3.7.4. The smaller stable fixed point a is called the ***refuge*** level of the budworm population, while the larger stable point c is the ***outbreak*** level. From the point of view of pest control, one would like to keep the population at a and away from c. The fate of the system is determined by the initial condition x_0; an outbreak occurs

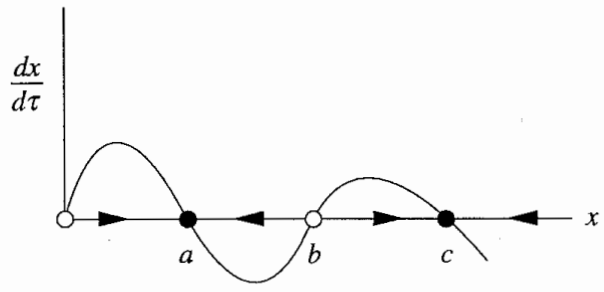


Figure 3.7.4

if and only if $x_0 > b$. In this sense the unstable equilibrium b plays the role of a ***threshold***.

An outbreak can also be triggered by a saddle-node bifurcation. If the parameters r and k drift in such a way that the fixed point a disappears, then the population will jump suddenly to the outbreak level c. The situation is made worse by the hysteresis effect—even if the parameters are restored to their values before the outbreak, the population will not drop back to the refuge level.

Calculating the Bifurcation Curves

Now we compute the curves in (k, r) space where the system undergoes saddle-node bifurcations. The calculation is somewhat harder than that in Section 3.6: we will not be able to write r explicitly as a function of k, for example. Instead, the bifurcation curves will be written in the ***parametric form*** $(k(x), r(x))$, where x runs through all positive values. (Please don't be confused by this traditional terminology—one would call x the "parameter" in these parametric equations, even though r and k are themselves parameters in a different sense.)

As discussed earlier, the condition for a saddle-node bifurcation is that the line $r(1 - x/k)$ intersects the curve $x/(1+x^2)$ tangentially. Thus we require *both*

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2} \tag{5}$$

and

$$\frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right]. \tag{6}$$

After differentiation, (6) reduces to

$$-\frac{r}{k}=\frac{1-x^2}{(1+x^2)^2}\ . \tag{7}$$

We substitute this expression for r/k into (5), which allows us to express r solely in terms of x. The result is

$$r=\frac{2x^3}{(1+x^2)^2}. \tag{8}$$

Then inserting (8) into (7) yields

$$k=\frac{2x^3}{x^2-1}. \tag{9}$$

The condition $k > 0$ implies that x must be restricted to the range $x > 1$.

Together (8) and (9) define the bifurcation curves. For each $x > 1$, we plot the

corresponding point $(k(x), r(x))$ in the (k, r) plane. The resulting curves are shown in Figure 3.7.5. (Exercise 3.7.2 deals with some of the analytical properties of these curves.)

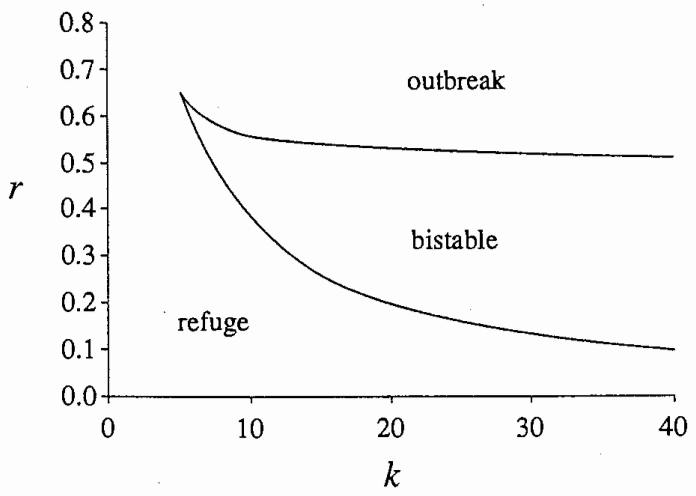


Figure 3.7.5

The different regions in Figure 3.7.5 are labeled according to the stable fixed points that exist. The refuge level a is the only stable state for low r, and the outbreak level c is the only stable state for large r. In the ***bistable*** region, both stable states exist.

The stability diagram is very similar to Figure 3.6.2. It too can be regarded as the projection of a cusp catastrophe surface, as schematically illustrated in Figure 3.7.6. You are hereby challenged to graph the surface accurately!

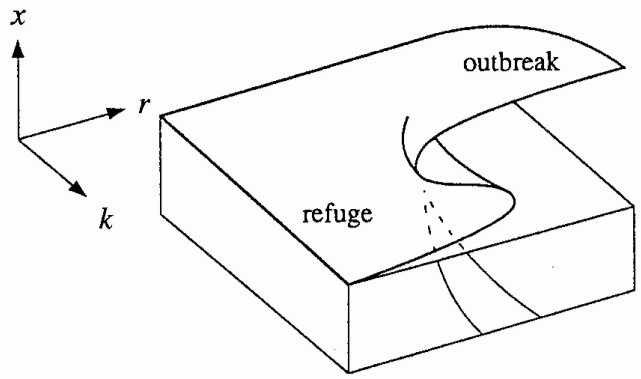


Figure 3.7.6

Comparison with Observations

Now we need to decide on biologically plausible values of the dimensionless groups $r = RA/B$ and $k = K/A$. A complication is that these parameters may drift

slowly as the condition of the forest changes. According to Ludwig et al. (1978), r increases as the forest grows, while k remains fixed.

They reason as follows: let S denote the average size of the trees, interpreted as the total surface area of the branches in a stand. Then the carrying capacity K should be proportional to the available foliage, so $K = K'S$. Similarly, the half-saturation parameter A in the predation term should be proportional to S; predators such as birds search *units of foliage*, not acres of forest, and so the relevant quantity A' must have the dimensions of budworms per unit of branch area. Hence $A = A'S$ and therefore

$$r = \frac{RA'}{B}S, \qquad k = \frac{K'}{A'}. \tag{10}$$

The experimental observations suggest that for a young forest, typically $k \approx 300$ and $r < 1/2$ so the parameters lie in the bistable region. The budworm population is kept down by the birds, which find it easy to search the small number of branches per acre. However, as the forest grows, S increases and therefore the point (k,r) drifts upward in parameter space toward the outbreak region of Figure 3.7.5. Ludwig et al. (1978) estimate that $r \approx 1$ for a fully mature forest, which lies dangerously in the outbreak region. After an outbreak occurs, the fir trees die and the forest is taken over by birch trees. But they are less efficient at using nutrients and eventually the fir trees come back—this recovery takes about 50–100 years (Murray 1989).

We conclude by mentioning some of the approximations in the model presented here. The tree dynamics have been neglected; see Ludwig et al. (1978) for a discussion of this longer time-scale behavior. We've also neglected the *spatial* distribution of budworms and their possible dispersal—see Ludwig et al. (1979) and Murray (1989) for treatments of this aspect of the problem.

EXERCISES FOR CHAPTER 3

3.1 Saddle-Node Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

3.1.1 $\dot{x} = 1 + rx + x^2$ **3.1.2** $\dot{x} = r - \cosh x$

3.1.3 $\dot{x} = r + x - \ln(1+x)$ **3.1.4** $\dot{x} = r + \frac{1}{2}x - x/(1+x)$

3.1.5 (Unusual bifurcations) In discussing the normal form of the saddle-node bi-

furcation, we mentioned the assumption that $a = \partial f/\partial r|_{(x^*, r_c)} \neq 0$. To see what can happen if $\partial f/\partial r|_{(x^*, r_c)} = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r.

a) $\dot{x} = r^2 - x^2$

b) $\dot{x} = r^2 + x^2$

3.2 Transcritical Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

3.2.1 $\dot{x} = rx + x^2$ **3.2.2** $\dot{x} = rx - \ln(1+x)$

3.2.3 $\dot{x} = x - rx(1-x)$ **3.2.4** $\dot{x} = x(r - e^x)$

3.2.5 (Chemical kinetics) Consider the chemical reaction system

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X \qquad X + B \xrightarrow{k_2} C.$$

This is a generalization of Exercise 2.3.2; the new feature is that X is used up in the production of C.

a) Assuming that both A and B are kept at constant concentrations a and b, show that the law of mass action leads to an equation of the form $\dot{x} = c_1 x - c_2 x^2$, where x is the concentration of X, and c_1 and c_2 are constants to be determined.

b) Show that $x^* = 0$ is stable when $k_2 b > k_1 a$, and explain why this makes sense chemically.

The next two exercises concern the normal form for the transcritical bifurcation. In Example 3.2.2, we showed how to reduce the dynamics near a transcritical bifurcation to the approximate form $\dot{X} = RX - X^2 + O(X^3)$. Our goal now is to show that the $O(X^3)$ terms can always be eliminated by a suitable nonlinear change of variables; in other words, the reduction to normal form can be made *exact*, not just approximate.

3.2.6 (Eliminating the cubic term) Consider the system $\dot{X} = RX - X^2 + aX^3 + O(X^4)$, where $R \neq 0$. We want to find a new variable x such that the system transforms into $\dot{x} = Rx - x^2 + O(x^4)$. This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped up to fourth order.

Let $x = X + bX^3 + O(X^4)$, where b will be chosen later to eliminate the cubic term in the differential equation for x. This is called a *near-identity transformation,* since x and X are practically equal; they differ by a tiny cubic term. (We

have skipped the quadratic term X^2, because it is not needed—you should check this later.) Now we need to rewrite the system in terms of x; this calculation requires a few steps.

a) Show that the near-identity transformation can be inverted to yield $X = x + cx^3 + O(x^4)$, and solve for c.

b) Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$, and substitute for X and $\dot{X}$ on the right-hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\dot{x} = Rx - x^2 + kx^3 + O(x^4)$, where k depends on a, b, and R.

c) Now the moment of triumph: Choose b so that $k = 0$.

d) Is is really necessary to make the assumption that $R \neq 0$? Explain.

3.2.7 (Eliminating any higher-order term) Now we generalize the method of the last exercise. Suppose we have managed to eliminate a number of higher-order terms, so that the system has been transformed into $\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1})$, where $n \geq 3$. Use the near-identity transformation $x = X + b_n X^n + O(X^{n+1})$ and the previous strategy to show that the system can be rewritten as $\dot{x} = Rx - x^2 + O(x^{n+1})$ for an appropriate choice of b_n. Thus we can eliminate as many higher-order terms as we like.

3.3 Laser Threshold

3.3.1 (An improved model of a laser) In the simple laser model considered in Section 3.3, we wrote an *algebraic* equation relating N, the number of excited atoms, to n, the number of laser photons. In more realistic models, this would be replaced by a *differential* equation. For instance, Milonni and Eberly (1988) show that after certain reasonable approximations, quantum mechanics leads to the system

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p.$$

Here G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, scattering, etc., f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign.

This two-dimensional system will be analyzed in Exercise 8.1.13. For now, let's convert it to a one-dimensional system, as follows.

a) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n. (This procedure is often called ***adiabatic elimination***, and one says that the evolution of $N(t)$ is *slaved* to that of $n(t)$. See Haken (1983).)

b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

c) What type of bifurcation occurs at the laser threshold p_c?
d) (Hard question) For what range of parameters is it valid to make the approximation used in (a)?

3.3.2 (Maxwell–Bloch equations) The Maxwell–Bloch equations provide an even more sophisticated model for a laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

where κ is the decay rate in the laser cavity due to beam transmission, γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is a pumping energy parameter. The parameter λ may be positive, negative, or zero; all the other parameters are positive.

These equations are similar to the Lorenz equations and can exhibit chaotic behavior (Haken 1983, Weiss and Vilaseca 1991). However, many practical lasers do not operate in the chaotic regime. In the simplest case $\gamma_1, \gamma_2 >> \kappa$; then P and D relax rapidly to steady values, and hence may be adiabatically eliminated, as follows.
a) Assuming $\dot{P} \approx 0$, $\dot{D} \approx 0$, express P and D in terms of E, and thereby derive a first-order equation for the evolution of E.
b) Find all the fixed points of the equation for E.
c) Draw the bifurcation diagram of E^* vs. λ. (Be sure to distinguish between stable and unstable branches.)

3.4 Pitchfork Bifurcation

In the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs. r.

3.4.1 $\dot{x} = rx + 4x^3$ **3.4.2** $\dot{x} = rx - \sinh x$

3.4.3 $\dot{x} = rx - 4x^3$ **3.4.4** $\dot{x} = x + \dfrac{rx}{1 + x^2}$

The next exercises are designed to test your ability to distinguish among the various types of bifurcations—it's easy to confuse them! In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

3.4.5 $\dot{x} = r - 3x^2$ **3.4.6** $\dot{x} = rx - \dfrac{x}{1+x}$

3.4.7 $\dot{x} = 5 - re^{-x^2}$ **3.4.8** $\dot{x} = rx - \dfrac{x}{1+x^2}$

3.4.9 $\dot{x} = x + \tanh(rx)$ **3.4.10** $\dot{x} = rx + \dfrac{x^3}{1+x^2}$

3.4.11 (An interesting bifurcation diagram) Consider the system $\dot{x} = rx - \sin x$.

a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.

b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?

c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.

d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.

e) Now classify all the bifurcations that occur as r decreases 0 to $-\infty$.

f) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

3.4.12 ("Quadfurcation") With tongue in cheek, we pointed out that the pitchfork bifurcation could be called a "trifurcation," since three branches of fixed points appear for $r > 0$. Can you construct an example of a "quadfurcation," in which $\dot{x} = f(x, r)$ has no fixed points for $r < 0$ and four branches of fixed points for $r > 0$? Extend your results to the case of an arbitrary number of branches, if possible.

3.4.13 (Computer work on bifurcation diagrams) For the vector fields below, use a computer to obtain a quantitatively accurate plot of the values of x^* vs. r, where $0 \le r \le 3$. In each case, there's an easy way to do this, and a harder way using the Newton-Raphson method.

a) $\dot{x} = r - x - e^{-x}$ b) $\dot{x} = 1 - x - e^{-rx}$

3.4.14 (Subcritical pitchfork) Consider the system $\dot{x} = rx + x^3 - x^5$, which exhibits a subcritical pitchfork bifurcation.

a) Find algebraic expressions for all the fixed points as r varies.

b) Sketch the vector fields as r varies. Be sure to indicate all the fixed points and their stability.

c) Calculate r_s, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

3.4.15 (First-order phase transition) Consider the potential $V(x)$ for the system $\dot{x} = rx + x^3 - x^5$. Calculate r_c, where r_c is defined by the condition that V has three equally deep wells, i.e., the values of V at the three local minima are equal.

(Note: In equilibrium statistical mechanics, one says that a *first-order phase transition* occurs at $r = r_c$. For this value of r, there is equal probability of finding the system in the state corresponding to any of the three minima. The freezing of water into ice is the most familiar example of a first-order phase transition.)

3.4.16 (Potentials) In parts (a)–(c), let $V(x)$ be the potential, in the sense that $\dot{x} = -dV/dx$. Sketch the potential as a function of r. Be sure to show all the qualitatively different cases, including bifurcation values of r.

a) (Saddle-node) $\dot{x} = r - x^2$

b) (Transcritical) $\dot{x} = rx - x^2$

c) (Subcritical pitchfork) $\dot{x} = rx + x^3 - x^5$

3.5 Overdamped Bead on a Rotating Hoop

3.5.1 Consider the bead on the rotating hoop discussed in Section 3.5. Explain in physical terms why the bead cannot have an equilibrium position with $\phi > \pi/2$.

3.5.2 Do the linear stability analysis for all the fixed points for Equation (3.5.7), and confirm that Figure 3.5.6 is correct.

3.5.3 Show that Equation (3.5.7) reduces to $\dfrac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^5)$ near $\phi = 0$. Find A and B.

3.5.4 (Bead on a horizontal wire) A bead of mass m is constrained to slide along a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is attached to the mass and to a support point a distance h from the wire (Figure 1).

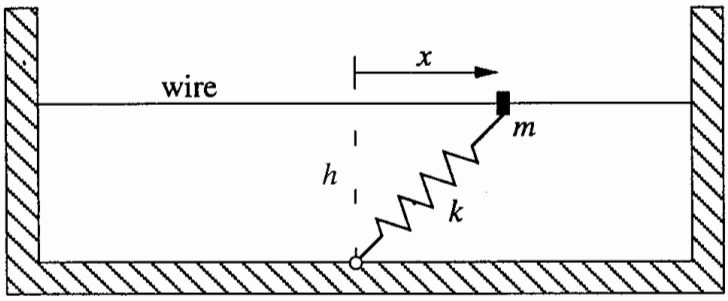


Figure 1

Finally, suppose that the motion of the bead is opposed by a viscous damping force $b\dot{x}$.

a) Write Newton's law for the motion of the bead.

b) Find all possible equilibria, i.e., fixed points, as functions of k, h, m, b, and L_0.

c) Suppose $m = 0$. Classify the stability of all the fixed points, and draw a bifurcation diagram.

d) If $m \neq 0$, how small does m have to be to be considered negligible? In what sense is it negligible?

3.5.5 (Time scale for the rapid transient) While considering the bead on the rotating hoop, we used phase plane analysis to show that the equation

$$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$$

has solutions that rapidly relax to the curve where $\frac{d\phi}{d\tau} = f(\phi)$.

a) Estimate the time scale T_{fast} for this rapid transient in terms of ε, and then express T_{fast} in terms of the original dimensional quantities m, g, r, ω, and b.

b) Rescale the original differential equation, using T_{fast} as the characteristic time scale, instead of $T_{slow} = b/mg$. Which terms in the equation are negligible on this time scale?

c) Show that $T_{fast} << T_{slow}$ if $\varepsilon << 1$. (In this sense, the time scales T_{fast} and T_{slow} are *widely separated*.)

3.5.6 (A model problem about singular limits) Consider the *linear* differential equation

$$\varepsilon\ddot{x} + \dot{x} + x = 0,$$

subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$.

a) Solve the problem analytically for all $\varepsilon > 0$.

b) Now suppose $\varepsilon << 1$. Show that there are two widely separated time scales in the problem, and estimate them in terms of ε.

c) Graph the solution $x(t)$ for $\varepsilon << 1$, and indicate the two time scales on the graph.

d) What do you conclude about the validity of replacing $\varepsilon\ddot{x} + \dot{x} + x = 0$ with its singular limit $\dot{x} + x = 0$?

e) Give two physical analogs of this problem, one involving a mechanical system, and another involving an electrical circuit. In each case, find the dimensionless combination of parameters corresponding to ε, and state the physical meaning of the limit $\varepsilon << 1$.

3.5.7 (Nondimensionalizing the logistic equation) Consider the logistic equation $\dot{N} = rN(1 - N/K)$, with initial condition $N(0) = N_0$.

a) This system has three dimensional parameters r, K, and N_0. Find the dimensions of each of these parameters.

b) Show that the system can be rewritten in the dimensionless form

$$\frac{dx}{d\tau} = x(1-x), \qquad x(0) = x_0$$

for appropriate choices of the dimensionless variables x, x_0, and τ.

c) Find a different nondimensionalization in terms of variables u and τ, where u is chosen such that the initial condition is always $u_0 = 1$.

d) Can you think of any advantage of one nondimensionalization over the other?

3.5.8 (Nondimensionalizing the subcritical pitchfork) The first-order system $\dot{u} = au + bu^3 - cu^5$, where $b, c > 0$, has a subcritical pitchfork bifurcation at $a = 0$. Show that this equation can be rewritten as

$$\frac{dx}{d\tau} = rx + x^3 - x^5$$

where $x = u/U$, $\tau = t/T$, and U, T, and r are to be determined in terms of a, b, and c.

3.6 Imperfect Bifurcations and Catastrophes

3.6.1 (Warm-up question about imperfect bifurcation) Does Figure 3.6.3b correspond to $h > 0$ or to $h < 0$?

3.6.2 (Imperfect transcritical bifurcation) Consider the system $\dot{x} = h + rx - x^2$. When $h = 0$, this system undergoes a transcritical bifurcation at $r = 0$. Our goal is to see how the bifurcation diagram of x^* vs. r is affected by the imperfection parameter h.

a) Plot the bifurcation diagram for $\dot{x} = h + rx - x^2$, for $h < 0$, $h = 0$, and $h > 0$.

b) Sketch the regions in the (r, h) plane that correspond to qualitatively different vector fields, and identify the bifurcations that occur on the boundaries of those regions.

c) Plot the potential $V(x)$ corresponding to all the different regions in the (r, h) plane.

3.6.3 (A perturbation to the supercritical pitchfork) Consider the system $\dot{x} = rx + ax^2 - x^3$, where $-\infty < a < \infty$. When $a = 0$, we have the normal form for the supercritical pitchfork. The goal of this exercise is to study the effects of the new parameter a.

a) For each a, there is a bifurcation diagram of x^* vs. r. As a varies, these bifurcation diagrams can undergo qualitative changes. Sketch all the qualitatively different bifurcation diagrams that can be obtained by varying a.

b) Summarize your results by plotting the regions in the (r, a) plane that correspond to qualitatively different classes of vector fields. Bifurcations occur on the boundaries of these regions; identify the types of bifurcations that occur.

3.6.4 (Imperfect saddle-node) What happens if you add a small imperfection to a system that has a saddle-node bifurcation?

3.6.5 (Mechanical example of imperfect bifurcation and catastrophe) Consider the bead on a tilted wire discussed at the end of Section 3.6.

a) Show that the equilibrium positions of the bead satisfy

$$mg\sin\theta = kx\left(1 - \frac{L_0}{\sqrt{x^2 + a^2}}\right).$$

b) Show that this equilibrium equation can be written in dimensionless form as

$$1 - \frac{h}{u} = \frac{R}{\sqrt{1+u^2}}$$

for appropriate choices of R, h, and u.

c) Give a graphical analysis of the dimensionless equation for the cases $R < 1$ and $R > 1$. How many equilibria can exist in each case?

d) Let $r = R - 1$. Show that the equilibrium equation reduces to $h + ru - \frac{1}{2}u^3 \approx 0$ for small r, h, and u.

e) Find an approximate formula for the saddle-node bifurcation curves in the limit of small r, h, and u.

f) Show that the *exact* equations for the bifurcation curves can be written in parametric form as

$$h(u) = -u^3, \qquad R(u) = (1+u^2)^{3/2},$$

where $-\infty < u < \infty$. (Hint: You may want to look at Section 3.7.) Check that this result reduces to the approximate result in part (d).

g) Give a numerically accurate plot of the bifurcation curves in the (r,h) plane.

h) Interpret your results physically, in terms of the original dimensional variables.

3.6.6 (Patterns in fluids) Ahlers (1989) gives a fascinating review of experiments on one-dimensional patterns in fluid systems. In many cases, the patterns first emerge via supercritical or subcritical pitchfork bifurcations from a spatially uniform state. Near the bifurcation, the dynamics of the amplitude of the patterns are given approximately by $\tau\dot{A} = \varepsilon A - gA^3$ in the supercritical case, or $\tau\dot{A} = \varepsilon A - gA^3 - kA^5$ in the subcritical case. Here $A(t)$ is the amplitude, τ is a typical time scale, and ε is a small dimensionless parameter that measures the distance from the bifurcation. The parameter $g > 0$ in the supercritical case, whereas $g < 0$ and $k > 0$ in the subcritical case. (In this context, the equation $\tau\dot{A} = \varepsilon A - gA^3$ is often called the *Landau equation.*)

a) Dubois and Bergé (1978) studied the supercritical bifurcation that arises in Rayleigh–Bénard convection, and showed experimentally that the steady-state amplitude depended on ε according to the power law $A^* \propto \varepsilon^{\beta}$, where $\beta = 0.50 \pm 0.01$. What does the Landau equation predict?

b) The equation $\tau\dot{A} = \varepsilon A - gA^3 - kA^5$ is said to undergo a *tricritical bifurcation*

when $g = 0$; this case is the borderline between supercritical and subcritical bifurcations. Find the relation between A^* and ε when $g = 0$.

c) In experiments on Taylor–Couette vortex flow, Aitta et al. (1985) were able to change the parameter g continuously from positive to negative by varying the aspect ratio of their experimental set-up. Assuming that the equation is modified to $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$, where $h > 0$ is a slight imperfection, sketch the bifurcation diagram of A^* vs. ε in the three cases $g > 0$, $g = 0$, and $g < 0$. Then look up the actual data in Aitta et al. (1985, Figure 2) or see Ahlers (1989, Figure 15).

d) In the experiments of part (c), the amplitude $A(t)$ was found to evolve toward a steady state in the manner shown in Figure 2 (redrawn from Ahlers (1989), Figure 18). The results are for the imperfect subcritical case $g < 0$, $h \neq 0$. In the experiments, the parameter ε was switched at $t = 0$ from a negative value to a positive value ε_f. In Figure 2, ε_f increases from the bottom to the top.

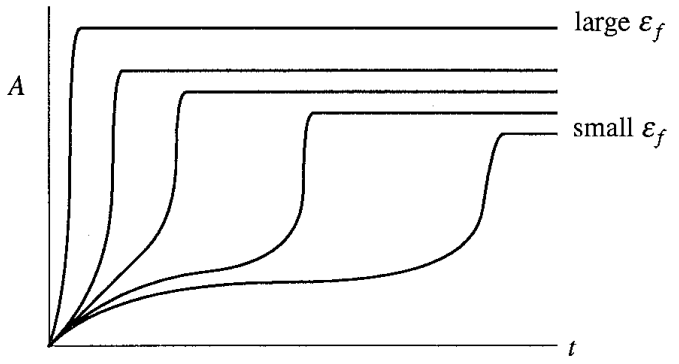


Figure 2

Explain intuitively why the curves have this strange shape. Why do the curves for large ε_f go almost straight up to their steady state, whereas the curves for small ε_f rise to a plateau before increasing sharply to their final level? (Hint: Graph $\dot{A}$ vs. A for different ε_f.)

3.6.7 (Simple model of a magnet) A magnet can be modeled as an enormous collection of electronic spins. In the simplest model, known as the ***Ising model***, the spins can point only up or down, and are assigned the values $S_i = \pm 1$, for $i = 1, \ldots, N >> 1$. For quantum mechanical reasons, the spins like to point in the same direction as their neighbors; on the other hand, the randomizing effects of temperature tend to disrupt any such alignment.

An important macroscopic property of the magnet is its average spin or *magnetization*

$$m = \left| \frac{1}{N} \sum_{i=1}^{N} S_i \right|.$$

At high temperature the spins point in random directions and so $m \approx 0$; the material is in the *paramagnetic* state. As the temperature is lowered, m remains near zero until a critical temperature T_c is reached. Then a ***phase transition*** occurs and the material spontaneously magnetizes. Now $m > 0$; we have a *ferromagnet*.

But the symmetry between up and down spins means that there are *two* possible ferromagnetic states. This symmetry can be broken by applying an external magnetic field h, which favors either the up or down direction. Then, in an approximation called *mean-field theory*, the equation governing the equilibrium value of m is

$$h = T \tanh^{-1} m - Jnm$$

where J and n are constants; $J > 0$ is the ferromagnetic coupling strength and n is the number of neighbors of each spin (Ma 1985, p. 459).

a) Analyze the solutions m^* of $h = T \tanh^{-1} m - Jnm$, using a graphical approach.

b) For the special case $h = 0$, find the critical temperature T_c at which a phase transition occurs.

3.7 Insect Outbreak

3.7.1 (Warm-up question about insect outbreak model) Show that the fixed point $x^* = 0$ is *always unstable* for Equation (3.7.3).

3.7.2 (Bifurcation curves for insect outbreak model)

a) Using Equations (3.7.8) and (3.7.9), sketch $r(x)$ and $k(x)$ vs. x. Determine the limiting behavior of $r(x)$ and $k(x)$ as $x \to 1$ and $x \to \infty$.

b) Find the exact values of r, k, and x at the cusp point shown in Figure 3.7.5.

3.7.3 (A model of a fishery) The equation $\dot{N} = rN(1 - \frac{N}{K}) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)

a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h,$$

for suitably defined dimensionless quantities x, τ, and h.

b) Plot the vector field for different values of h.

c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.

d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model—the population can become nega-

tive! A better model would have a fixed point at zero population for all values of H. See the next exercise for such an improvement.

3.7.4 (Improved model of a fishery) A refinement of the model in the last exercise is

$$\dot{N} = rN\left(1-\frac{N}{K}\right) - H\frac{N}{A+N}$$

where $H>0$ and $A>0$. This model is more realistic in two respects: it has a fixed point at $N=0$ for all values of the parameters, and the rate at which fish are caught decreases with N. This is plausible—when fewer fish are available, it is harder to find them and so the daily catch drops.

a) Give a biological interpretation of the parameter A; what does it measure?

b) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1-x) - h\frac{x}{a+x},$$

for suitably defined dimensionless quantities x, τ, a, and h.

c) Show that the system can have one, two, or three fixed points, depending on the values of a and h. Classify the stability of the fixed points in each case.

d) Analyze the dynamics near $x=0$ and show that a bifurcation occurs when $h=a$. What type of bifurcation is it?

e) Show that another bifurcation occurs when $h=\frac{1}{4}(a+1)^2$, for $a<a_c$, where a_c is to be determined. Classify this bifurcation.

f) Plot the stability diagram of the system in (a,h) parameter space. Can hysteresis occur in any of the stability regions?

3.7.5 (A biochemical switch) Zebra stripes and butterfly wing patterns are two of the most spectacular examples of biological pattern formation. Explaining the development of these patterns is one of the outstanding problems of biology; see Murray (1989) for an excellent review of our current knowledge.

As one ingredient in a model of pattern formation, Lewis et al. (1977) considered a simple example of a biochemical switch, in which a gene G is activated by a biochemical signal substance S. For example, the gene may normally be inactive but can be "switched on" to produce a pigment or other gene product when the concentration of S exceeds a certain threshold. Let $g(t)$ denote the concentration of the gene product, and assume that the concentration s_0 of S is fixed. The model is

$$\dot{g} = k_1 s_0 - k_2 g + \frac{k_3 g^2}{k_4^{\,2} + g^2}$$

where the k's are positive constants. The production of g is stimulated by s_0 at a

rate k_1, and by an *autocatalytic* or positive feedback process (the nonlinear term). There is also a linear degradation of g at a rate k_2.

a) Show that the system can be put in the dimensionless form

$$\frac{dx}{d\tau} = s - rx + \frac{x^2}{1+x^2}$$

where $r > 0$ and $s \geq 0$ are dimensionless groups.

b) Show that if $s = 0$, there are two positive fixed points x^* if $r < r_c$, where r_c is to be determined.

c) Assume that initially there is no gene product, i.e., $g(0) = 0$, and suppose s is slowly increased from zero (the activating signal is turned on); what happens to $g(t)$? What happens if s then goes back to zero? Does the gene turn off again?

d) Find parametric equations for the bifurcation curves in (r, s) space, and classify the bifurcations that occur.

e) Use the computer to give a quantitatively accurate plot of the stability diagram in (r, s) space.

For further discussion of this model, see Lewis et al. (1977); Edelstein–Keshet (1988), Section 7.5; or Murray (1989), Chapter 15.

3.7.6 (Model of an epidemic) In pioneering work in epidemiology, Kermack and McKendrick (1927) proposed the following simple model for the evolution of an epidemic. Suppose that the population can be divided into three classes: $x(t)$ = number of healthy people; $y(t)$ = number of sick people; $z(t)$ = number of dead people. Assume that the total population remains constant in size, except for deaths due to the epidemic. (That is, the epidemic evolves so rapidly that we can ignore the slower changes in the populations due to births, emigration, or deaths by other causes.)

Then the model is

$$\begin{aligned} \dot{x} &= -kxy \\ \dot{y} &= kxy - \ell y \\ \dot{z} &= \ell y \end{aligned}$$

where k and ℓ are positive constants. The equations are based on two assumptions:

(i) Healthy people get sick at a rate proportional to the product of x and y. This would be true if healthy and sick people encounter each other at a rate proportional to their numbers, and if there were a constant probability that each such encounter would lead to transmission of the disease.

(ii) Sick people die at a constant rate ℓ.

The goal of this exercise is to reduce the model, which is a *third-order system*, to a first-order system that can analyzed by our methods. (In Chapter 6 we will see

a simpler analysis.)

a) Show that $x + y + z = N$, where N is constant.

b) Use the $\dot{x}$ and $\dot{z}$ equation to show that $x(t) = x_0 \exp(-kz(t)/\ell)$, where $x_0 = x(0)$.

c) Show that z satisfies the first-order equation $\dot{z} = \ell\left[N - z - x_0 \exp(-kz/\ell)\right]$.

d) Show that this equation can be nondimensionalized to

$$\frac{du}{d\tau} = a - bu - e^{-u}$$

by an appropriate rescaling.

e) Show that $a \geq 1$ and $b > 0$.

f) Determine the number of fixed points u^* and classify their stability.

g) Show that the maximum of $\dot{u}(t)$ occurs at the same time as the maximum of both $\dot{z}(t)$ and $y(t)$. (This time is called the ***peak*** of the epidemic, denoted t_{peak}. At this time, there are more sick people and a higher daily death rate than at any other time.)

h) Show that if $b < 1$, then $\dot{u}(t)$ is increasing at $t = 0$ and reaches its maximum at some time $t_{\text{peak}} > 0$. Thus things get worse before they get better. (The term ***epidemic*** is reserved for this case.) Show that $\dot{u}(t)$ eventually decreases to 0.

i) On the other hand, show that $t_{\text{peak}} = 0$ if $b > 1$. (Hence no epidemic occurs if $b > 1$.)

j) The condition $b = 1$ is the *threshold* condition for an epidemic to occur. Can you give a biological interpretation of this condition?

k) Kermack and McKendrick showed that their model gave a good fit to data from the Bombay plague of 1906. How would you improve the model to make it more appropriate for AIDS? Which assumptions need revising?

For an introduction to models of epidemics, see Murray (1989), Chapter 19, or Edelstein–Keshet (1988). Models of AIDS are discussed by Murray (1989) and May and Anderson (1987). An excellent review and commentary on the Kermack–McKendrick papers is given by Anderson (1991).