2

FLOWS ON THE LINE

2.0 Introduction

In Chapter 1, we introduced the general system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \ldots, x_n) \end{aligned}$$

and mentioned that its solutions could be visualized as trajectories flowing through an n-dimensional phase space with coordinates $(x_1, \ldots, x_n)$. At the moment, this idea probably strikes you as a mind-bending abstraction. So let's start slowly, beginning here on earth with the simple case $n = 1$. Then we get a single equation of the form

$$\dot{x} = f(x).$$

Here $x(t)$ is a real-valued function of time t, and $f(x)$ is a smooth real-valued function of x. We'll call such equations ***one-dimensional*** or ***first-order systems***.

Before there's any chance of confusion, let's dispense with two fussy points of terminology:

1. The word *system* is being used here in the sense of a dynamical system, not in the classical sense of a collection of two or more equations. Thus a single equation can be a "system."
2. We do not allow f to depend explicitly on time. Time-dependent or "nonautonomous" equations of the form $\dot{x} = f(x,t)$ are more complicated, because one needs *two* pieces of information, x and t, to predict the future state of the system. Thus $\dot{x} = f(x,t)$ should really be regarded as a *two-dimensional* or *second-order* system, and will therefore be discussed later in the book.

2.1 A Geometric Way of Thinking

Pictures are often more helpful than formulas for analyzing nonlinear systems. Here we illustrate this point by a simple example. Along the way we will introduce one of the most basic techniques of dynamics: *interpreting a differential equation as a vector field.*

Consider the following nonlinear differential equation:

$$\dot{x} = \sin x. \tag{1}$$

To emphasize our point about formulas versus pictures, we have chosen one of the few nonlinear equations that can be solved in closed form. We separate the variables and then integrate:

$$dt = \frac{dx}{\sin x},$$

which implies

$$t = \int \csc x \, dx$$

$$= -\ln|\csc x + \cot x| + C.$$

To evaluate the constant C, suppose that $x = x_0$ at $t = 0$. Then $C = \ln|\csc x_0 + \cot x_0|$. Hence the solution is

$$t = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|. \tag{2}$$

This result is exact, but a headache to interpret. For example, can you answer the following questions?

1. Suppose $x_0 = \pi/4$; describe the qualitative features of the solution $x(t)$ for all $t > 0$. In particular, what happens as $t \to \infty$?
2. For an *arbitrary* initial condition x_0, what is the behavior of $x(t)$ as $t \to \infty$?

Think about these questions for a while, to see that formula (2) is not transparent.

In contrast, a graphical analysis of (1) is clear and simple, as shown in Figure 2.1.1. We think of t as time, x as the position of an imaginary particle moving along the real line, and $\dot{x}$ as the velocity of that particle. Then the differential equation $\dot{x} = \sin x$ represents a ***vector field*** on the line: it dictates the velocity vector $\dot{x}$ at each x. To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the x-axis to indicate the corresponding velocity vector at each x. The arrows point to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$.

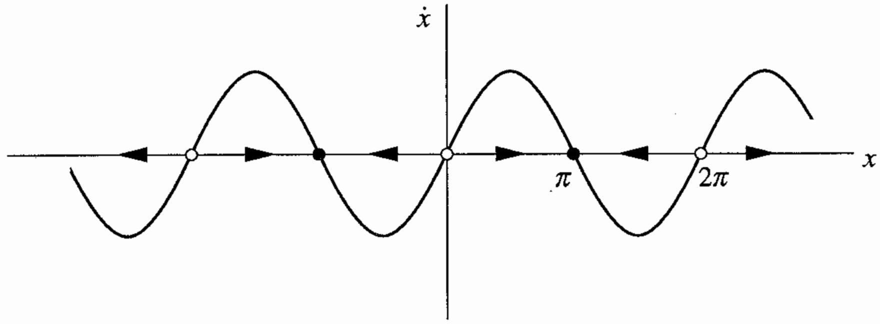

Figure 2.1.1

Here's a more physical way to think about the vector field: imagine that fluid is flowing steadily along the x-axis with a velocity that varies from place to place, according to the rule $\dot{x} = \sin x$. As shown in Figure 2.1.1, the ***flow*** is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At points where $\dot{x} = 0$, there is no flow; such points are therefore called ***fixed points***. You can see that there are two kinds of fixed points in Figure 2.1.1: solid black dots represent ***stable*** fixed points (often called *attractors* or *sinks*, because the flow is toward them) and open circles represent ***unstable*** fixed points (also known as *repellers* or *sources*).

Armed with this picture, we can now easily understand the solutions to the differential equation $\dot{x} = \sin x$. We just start our imaginary particle at x_0 and watch how it is carried along by the flow.

This approach allows us to answer the questions above as follows:

1. Figure 2.1.1 shows that a particle starting at $x_0 = \pi/4$ moves to the right faster and faster until it crosses $x = \pi/2$ (where $\sin x$ reaches its maximum). Then the particle starts slowing down and eventually approaches the stable fixed point $x = \pi$ from the left. Thus, the qualitative form of the solution is as shown in Figure 2.1.2.

   Note that the curve is concave up at first, and then concave down; this corresponds to the initial acceleration for $x < \pi/2$, followed by the deceleration toward $x = \pi$.

2. The same reasoning applies to any initial condition x_0. Figure 2.1.1 shows that if $\dot{x} > 0$ initially, the particle heads to the right and asymptotically approaches the nearest stable fixed point. Similarly, if $\dot{x} < 0$ initially, the particle approaches the nearest stable fixed point to its left. If $\dot{x} = 0$, then x remains constant. The qualitative form of the solution for any initial condition is sketched in Figure 2.1.3.

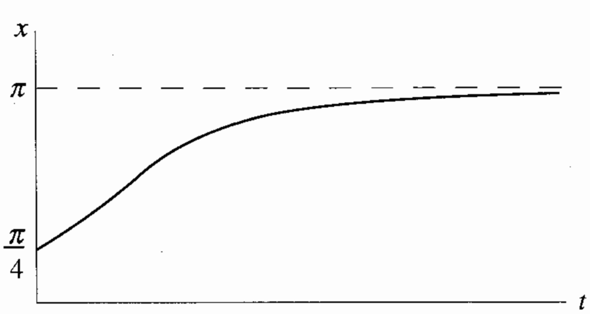

Figure 2.1.2

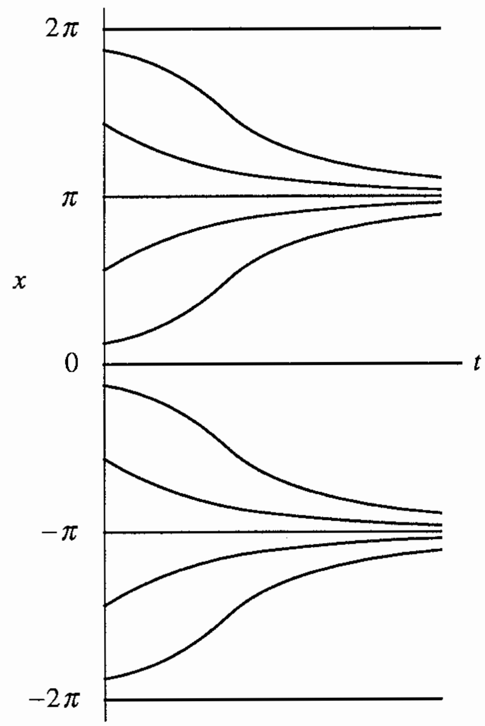

Figure 2.1.3

In all honesty, we should admit that a picture can't tell us certain *quantitative* things: for instance, we don't know the time at which the speed $|\dot{x}|$ is greatest. But in many cases *qualitative* information is what we care about, and then pictures are fine.

2.2 Fixed Points and Stability

The ideas developed in the last section can be extended to any one-dimensional system $\dot{x} = f(x)$. We just need to draw the graph of $f(x)$ and then use it to sketch the vector field on the real line (the x-axis in Figure 2.2.1).

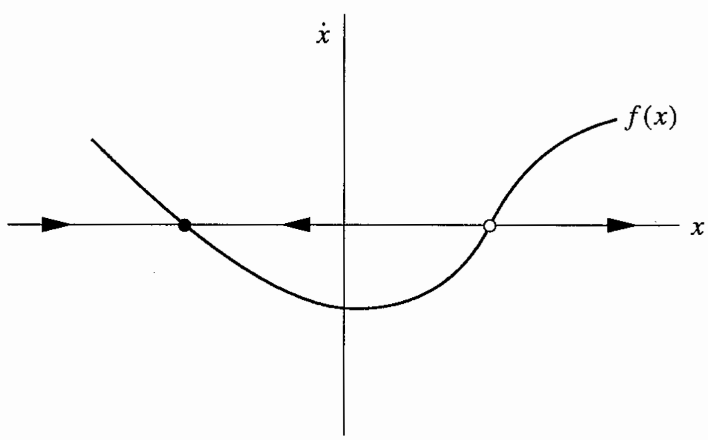

Figure 2.2.1

As before, we imagine that a fluid is flowing along the real line with a local velocity $f(x)$. This imaginary fluid is called the phase fluid, and the real line is the phase space. The flow is to the right where $f(x) > 0$ and to the left where $f(x) < 0$. To find the solution to $\dot{x} = f(x)$ starting from an arbitrary initial condition x_0, we place an imaginary particle (known as a ***phase point***) at x_0 and watch how it is carried along by the flow. As time goes on, the phase point moves along the x-axis according to some function $x(t)$. This function is called the ***trajectory*** based at x_0, and it represents the solution of the differential equation starting from the initial condition x_0. A picture like Figure 2.2.1, which shows all the qualitatively different trajectories of the system, is called a ***phase portrait.***

The appearance of the phase portrait is controlled by the fixed points x^*, defined by $f(x^*) = 0$; they correspond to stagnation points of the flow. In Figure 2.2.1, the solid black dot is a stable fixed point (the local flow is toward it) and the open dot is an unstable fixed point (the flow is away from it).

In terms of the original differential equation, fixed points represent ***equilibrium*** solutions (sometimes called steady, constant, or rest solutions, since if $x = x^*$ initially, then $x(t) = x^*$ for all time). An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time. Thus stable equilibria are represented geometrically by stable fixed points. Conversely, unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points.

EXAMPLE 2.2.1:

Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

Solution: Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field (Figure 2.2.2). The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable. ■

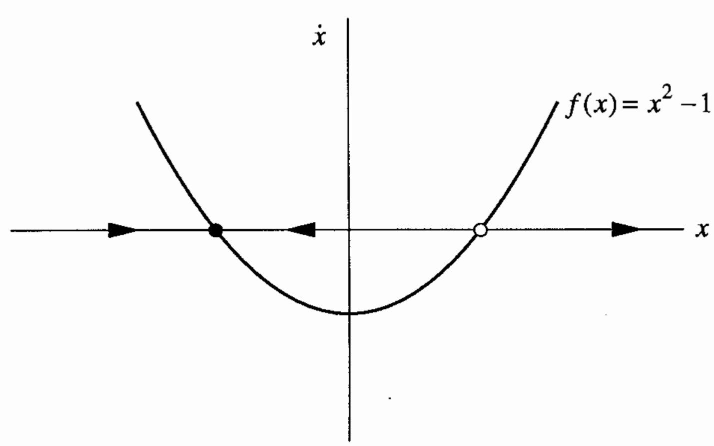

Figure 2.2.2

Note that the definition of stable equilibrium is based on *small* disturbances; certain large disturbances may fail to decay. In Example 2.2.1, all small disturbances to $x^* = -1$ will decay, but a large disturbance that sends x to the right of $x = 1$ will *not* decay—in fact, the phase point will be repelled out to $+\infty$. To emphasize this aspect of stability, we sometimes say that $x^* = -1$ is ***locally stable***, but not globally stable.

EXAMPLE 2.2.2:

Consider the electrical circuit shown in Figure 2.2.3. A resistor R and a capacitor C are in series with a battery of constant dc voltage V_0. Suppose that the switch is closed at $t = 0$, and that there is no charge on the capacitor initially. Let $Q(t)$ denote the charge on the capacitor at time $t \geq 0$. Sketch the graph of $Q(t)$.

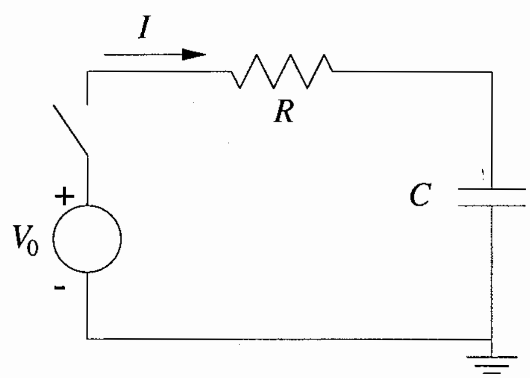

Figure 2.2.3

Solution: This type of circuit problem is probably familiar to you. It is governed by linear equations and can be solved analytically, but we prefer to illustrate the geometric approach.

First we write the circuit equations. As we go around the circuit, the total voltage drop must equal zero; hence $-V_0 + RI + Q/C = 0$, where I is the current flowing through the resistor. This current causes charge to accumulate on the capacitor at a rate $\dot{Q} = I$. Hence

$$-V_0 + R\dot{Q} + Q/C = 0 \quad \text{or}$$

$$\dot{Q} = f(Q) = \frac{V_0}{R} - \frac{Q}{RC}.$$

The graph of $f(Q)$ is a straight line with a negative slope (Figure 2.2.4). The corresponding vector field has a fixed point where $f(Q) = 0$, which occurs at $Q^* = CV_0$. The flow is to the right where $f(Q) > 0$ and to the left where $f(Q) < 0$. Thus the flow is always toward Q^*—it is a *stable* fixed point. In fact, it is ***globally stable***, in the sense that it is approached from *all* initial conditions.

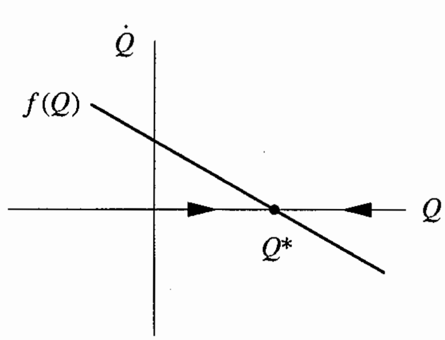

Figure 2.2.4

To sketch $Q(t)$, we start a phase point at the origin of Figure 2.2.4 and imagine how it would move. The flow carries the phase point monotonically toward Q^*. Its speed

$\dot{Q}$ decreases linearly as it approaches the fixed point; therefore $Q(t)$ is increasing and concave down, as shown in Figure 2.2.5. ■

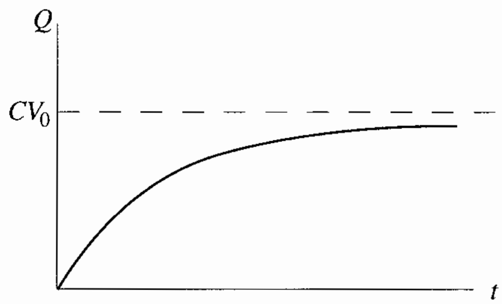

Figure 2.2.5

EXAMPLE 2.2.3:

Sketch the phase portrait corresponding to $\dot{x} = x - \cos x$, and determine the stability of all the fixed points.

Solution: One approach would be to plot the function $f(x) = x - \cos x$ and then sketch the associated vector field. This method is valid, but it requires you to figure out what the graph of $x - \cos x$ looks like.

There's an easier solution, which exploits the fact that we know how to graph $y = x$ and $y = \cos x$ *separately*. We plot both graphs on the same axes and then observe that they intersect in exactly one point (Figure 2.2.6).

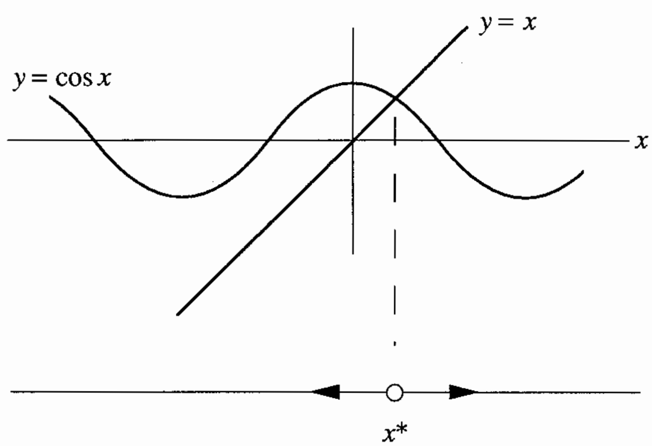

Figure 2.2.6

This intersection corresponds to a fixed point, since $x^* = \cos x^*$ and therefore $f(x^*) = 0$. Moreover, when the line lies above the cosine curve, we have $x > \cos x$ and so $\dot{x} > 0$: the flow is to the right. Similarly, the flow is to the left where the line is below the cosine curve. Hence x^* is the only fixed point, and it is unstable. Note that we can classify the stability of x^*, even though we don't have a formula for x^* itself! ■

2.3 Population Growth

The simplest model for the growth of a population of organisms is $\dot{N} = rN$, where $N(t)$ is the population at time t, and $r > 0$ is the growth rate. This model

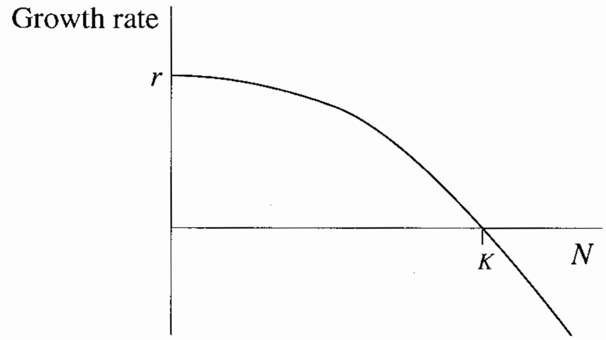

Figure 2.3.1

predicts exponential growth: $N(t) = N_0 e^{rt}$, where N_0 is the population at $t = 0$.

Of course such exponential growth cannot go on forever. To model the effects of overcrowding and limited resources, population biologists and demographers often assume that the per capita growth rate $\dot{N}/N$ decreases when N becomes sufficiently large, as shown in Figure 2.3.1. For small N, the growth rate equals r, just as before. However, for populations larger than a certain ***carrying capacity*** K, the growth rate actually becomes negative; the death rate is higher than the birth rate.

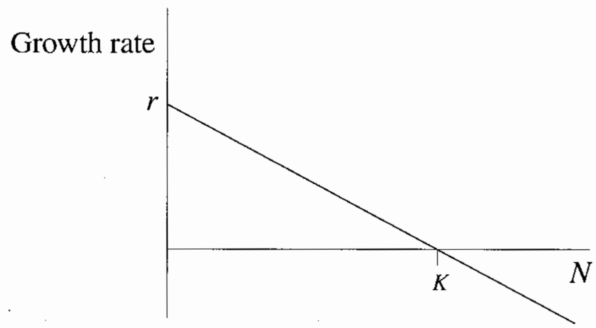

Figure 2.3.2

A mathematically convenient way to incorporate these ideas is to assume that the per capita growth rate $\dot{N}/N$ decreases *linearly* with N (Figure 2.3.2).

This leads to the ***logistic equation***

$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$

first suggested to describe the growth of human populations by Verhulst in 1838. This equation can be solved analytically (Exercise 2.3.1) but once again we prefer a graphical approach. We plot $\dot{N}$ versus N to see what the vector field looks like. Note that we plot only $N \geq 0$, since it makes no sense to think about a negative population (Figure 2.3.3). Fixed points occur at $N^* = 0$ and $N^* = K$, as found by setting $\dot{N} = 0$ and solving for N. By looking at the flow in Figure 2.3.3, we see that $N^* = 0$ is an unstable fixed point and $N^* = K$ is a stable fixed point. In biological terms, $N = 0$ is an unstable equilibrium: a small population will grow exponentially fast and run away from $N = 0$. On the other hand, if N is disturbed slightly from K, the disturbance will decay monotonically and $N(t) \to K$ as $t \to \infty$.

In fact, Figure 2.3.3 shows that if we start a phase point at *any* $N_0 > 0$, it will always flow toward $N = K$. Hence *the population always approaches the carrying capacity.*

The only exception is if $N_0 = 0$; then there's nobody around to start reproducing, and so $N = 0$ for all time. (The model does not allow for spontaneous generation!)

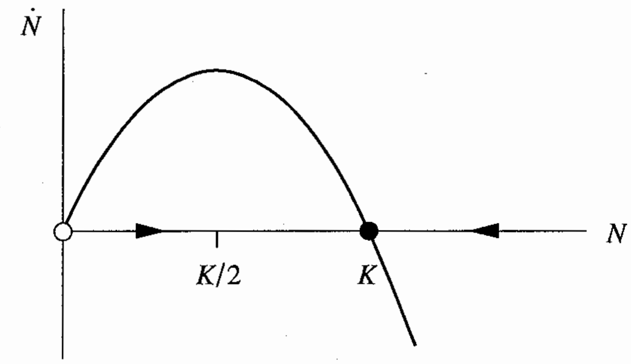

Figure 2.3.3

Figure 2.3.3 also allows us to deduce the qualitative shape of the solutions. For example, if $N_0 < K/2$, the phase point moves faster and faster until it crosses $N = K/2$, where the parabola in Figure 2.3.3 reaches its maximum. Then the phase point slows down and eventually creeps toward $N = K$. In biological terms, this means that the population initially grows in an accelerating fashion, and the graph of $N(t)$ is concave up. But after $N = K/2$, the derivative $\dot{N}$ begins to decrease, and so $N(t)$ is concave down as it asymptotes to the horizontal line $N = K$ (Figure 2.3.4). Thus the graph of $N(t)$ is S-shaped or *sigmoid* for $N_0 < K/2$.

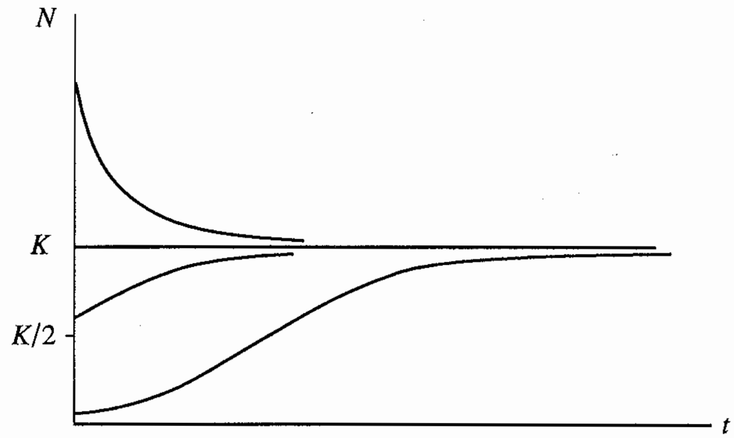

Figure 2.3.4

Something qualitatively different occurs if the initial condition N_0 lies between $K/2$ and K; now the solutions are decelerating from the start. Hence these solutions are concave down for all t. If the population initially exceeds the carrying capacity ($N_0 > K$), then $N(t)$ decreases toward $N = K$ and is concave up. Finally, if $N_0 = 0$ or $N_0 = K$, then the population stays constant.

Critique of the Logistic Model

Before leaving this example, we should make a few comments about the biological validity of the logistic equation. The algebraic form of the model is not to be taken literally. The model should really be regarded as a metaphor for populations that have a tendency to grow from zero population up to some carrying capacity K.

Originally a much stricter interpretation was proposed, and the model was argued to be a universal law of growth (Pearl 1927). The logistic equation was tested in laboratory experiments in which colonies of bacteria, yeast, or other simple organisms were grown in conditions of constant climate, food supply, and absence of predators. For a good review of this literature, see Krebs (1972, pp. 190–200). These experiments often yielded sigmoid growth curves, in some cases with an impressive match to the logistic predictions.

On the other hand, the agreement was much worse for fruit flies, flour beetles, and other organisms that have complex life cycles, involving eggs, larvae, pupae, and adults. In these organisms, the predicted asymptotic approach to a steady carrying capacity was never observed—instead the populations exhibited large, persistent fluctuations after an initial period of logistic growth. See Krebs (1972) for a discussion of the possible causes of these fluctuations, including age structure and time-delayed effects of overcrowding in the population.

For further reading on population biology, see Pielou (1969) or May (1981). Edelstein–Keshet (1988) and Murray (1989) are excellent textbooks on mathematical biology in general.

2.4 Linear Stability Analysis

So far we have relied on graphical methods to determine the stability of fixed points. Frequently one would like to have a more quantitative measure of stability, such as the rate of decay to a stable fixed point. This sort of information may be obtained by *linearizing* about a fixed point, as we now explain.

Let x^* be a fixed point, and let $\eta(t) = x(t) - x^*$ be a small perturbation away from x^*. To see whether the perturbation grows or decays, we derive a differential equation for η. Differentiation yields

$$\dot{\eta} = \tfrac{d}{dt}(x - x^*) = \dot{x},$$

since x^* is constant. Thus $\dot{\eta} = \dot{x} = f(x) = f(x^* + \eta)$. Now using Taylor's expansion we obtain

$$f(x^* + \eta) = f(x^*) + \eta f'(x^*) + O(\eta^2),$$

where $O(\eta^2)$ denotes quadratically small terms in η. Finally, note that $f(x^*) = 0$ since x^* is a fixed point. Hence

$$\dot{\eta} = \eta f'(x^*) + O(\eta^2).$$

Now if $f'(x^*) \neq 0$, the $O(\eta^2)$ terms are negligible and we may write the approximation

$$\dot{\eta} \approx \eta f'(x^*) .$$

This is a linear equation in η, and is called the ***linearization about*** x^*. It shows that *the perturbation* $\eta(t)$ *grows exponentially if* $f'(x^*) > 0$ *and decays if* $f'(x^*) < 0$. If $f'(x^*) = 0$, the $O(\eta^2)$ terms are not negligible and a nonlinear analysis is needed to determine stability, as discussed in Example 2.4.3 below.

The upshot is that the slope $f'(x^*)$ at the fixed point determines its stability. If you look back at the earlier examples, you'll see that the slope was always negative at a stable fixed point. The importance of the *sign* of $f'(x^*)$ was clear from our graphical approach; the new feature is that now we have a measure of *how* stable a fixed point is—that's determined by the *magnitude* of $f'(x^*)$. This magnitude plays the role of an exponential growth or decay rate. Its reciprocal $1/|f'(x^*)|$ is a ***characteristic time scale***; it determines the time required for $x(t)$ to vary significantly in the neighborhood of x^*.

EXAMPLE 2.4.1:

Using linear stability analysis, determine the stability of the fixed points for $\dot{x} = \sin x$.

Solution: The fixed points occur where $f(x) = \sin x = 0$. Thus $x^* = k\pi$, where k is an integer. Then

$$f'(x^*) = \cos k\pi = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd}. \end{cases}$$

Hence x^* is unstable if k is even and stable if k is odd. This agrees with the results shown in Figure 2.1.1. ■

EXAMPLE 2.4.2:

Classify the fixed points of the logistic equation, using linear stability analysis, and find the characteristic time scale in each case.

Solution: Here $f(N) = rN\left(1 - \frac{N}{K}\right)$, with fixed points $N^* = 0$ and $N^* = K$. Then $f'(N) = r - \frac{2rN}{K}$ and so $f'(0) = r$ and $f'(K) = -r$. Hence $N^* = 0$ is unstable and $N^* = K$ is stable, as found earlier by graphical arguments. In either case, the characteristic time scale is $1/|f'(N^*)| = 1/r$. ■

EXAMPLE 2.4.3:

What can be said about the stability of a fixed point when $f'(x^*) = 0$?

Solution: Nothing can be said in general. The stability is best determined on a case-by-case basis, using graphical methods. Consider the following examples:

(a) $\dot{x} = -x^3$ (b) $\dot{x} = x^3$ (c) $\dot{x} = x^2$ (d) $\dot{x} = 0$

Each of these systems has a fixed point $x^* = 0$ with $f'(x^*) = 0$. However the stability is different in each case. Figure 2.4.1 shows that (a) is stable and (b) is unstable. Case (c) is a hybrid case we'll call ***half-stable***, since the fixed point is attracting from the left and repelling from the right. We therefore indicate this type of fixed point by a half-filled circle. Case (d) is a whole line of fixed points; perturbations neither grow nor decay.

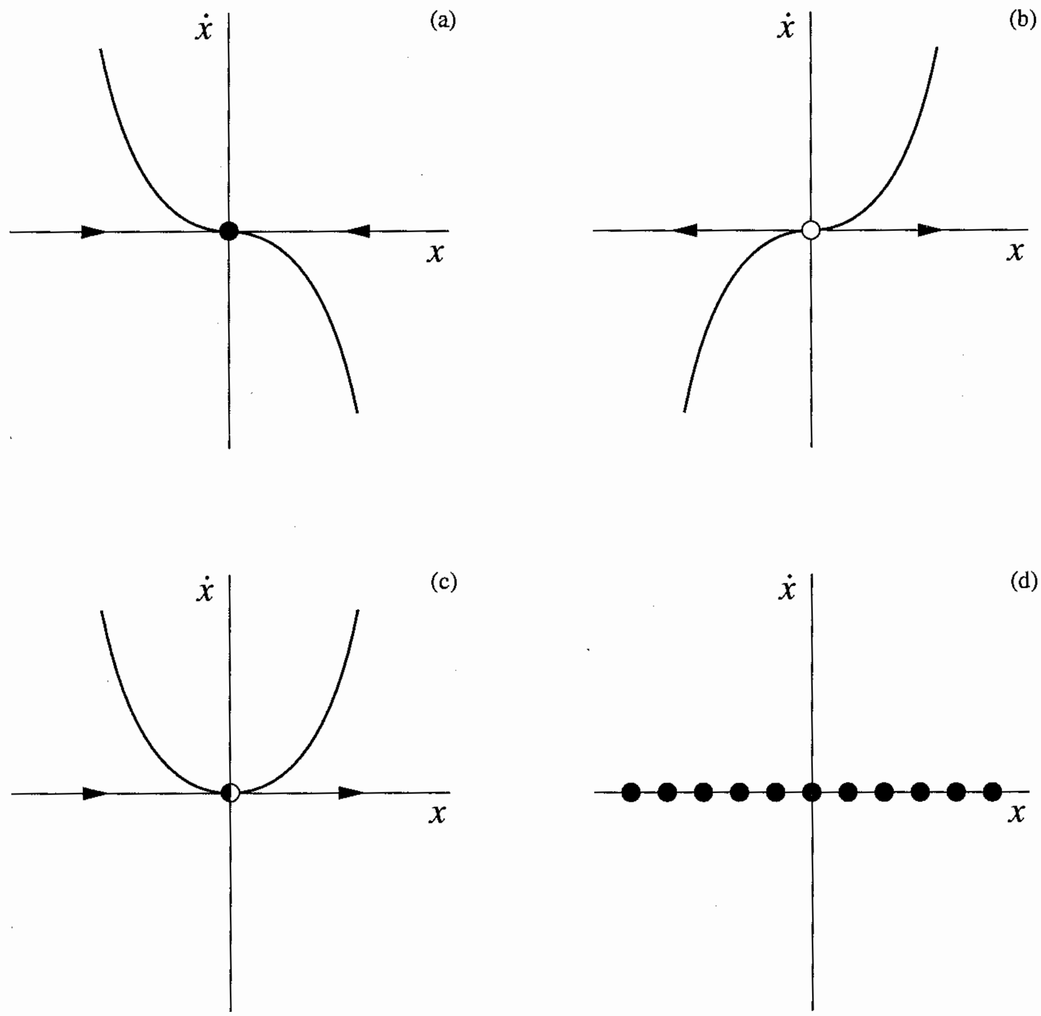

Figure 2.4.1

These examples may seem artificial, but we will see that they arise naturally in the context of *bifurcations*—more about that later. ■

2.5 Existence and Uniqueness

Our treatment of vector fields has been very informal. In particular, we have taken a cavalier attitude toward questions of existence and uniqueness of solutions to the system $\dot{x} = f(x)$. That's in keeping with the "applied" spirit of this book. Nevertheless, we should be aware of what can go wrong in pathological cases.

EXAMPLE 2.5.1:

Show that the solution to $\dot{x} = x^{1/3}$ starting from $x_0 = 0$ is *not* unique.

Solution: The point $x = 0$ is a fixed point, so one obvious solution is $x(t) = 0$ for all t. The surprising fact is that there is *another* solution. To find it we separate variables and integrate:

$$\int x^{-1/3} dx = \int dt$$

so $\frac{3}{2} x^{2/3} = t + C$. Imposing the initial condition $x(0) = 0$ yields $C = 0$. Hence $x(t) = \left(\frac{2}{3} t\right)^{3/2}$ is also a solution! ■

When uniqueness fails, our geometric approach collapses because the phase point doesn't know how to move; if a phase point were started at the origin, would it stay there or would it move according to $x(t) = \left(\frac{2}{3} t\right)^{3/2}$? (Or as my friends in elementary school used to say when discussing the problem of the irresistible force and the immovable object, perhaps the phase point would explode!)

Actually, the situation in Example 2.5.1 is even worse than we've let on—there are *infinitely* many solutions starting from the same initial condition (Exercise 2.5.4).

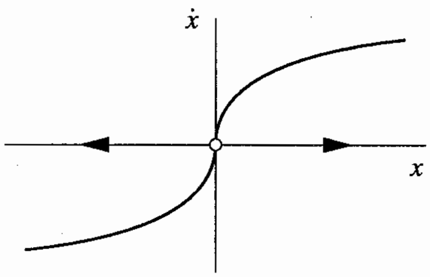

Figure 2.5.1

What's the source of the non-uniqueness? A hint comes from looking at the vector field (Figure 2.5.1). We see that the fixed point $x^* = 0$ is *very* unstable—the slope $f'(0)$ is infinite.

Chastened by this example, we state a theorem that provides sufficient conditions for existence and uniqueness of solutions to $\dot{x} = f(x)$.

Existence and Uniqueness Theorem: Consider the initial value problem

$$\dot{x} = f(x), \qquad x(0) = x_0.$$

Suppose that $f(x)$ and $f'(x)$ are continuous on an open interval R of the x-axis, and suppose that x_0 is a point in R. Then the initial value problem has a solution $x(t)$ on some time interval $(-\tau, \tau)$ about $t = 0$, and the solution is unique.

For proofs of the existence and uniqueness theorem, see Borrelli and Coleman (1987), Lin and Segel (1988), or virtually any text on ordinary differential equations.

This theorem says that *if* $f(x)$ *is smooth enough,* then solutions exist and are unique. Even so, there's no guarantee that solutions exist forever, as shown by the next example.

EXAMPLE 2.5.2:

Discuss the existence and uniqueness of solutions to the initial value problem $\dot{x} = 1 + x^2$, $x(0) = x_0$. Do solutions exist for all time?

Solution: Here $f(x) = 1 + x^2$. This function is continuous and has a continuous derivative for all x. Hence the theorem tells us that solutions exist and are unique for any initial condition x_0. But *the theorem does not say that the solutions exist for all time;* they are only guaranteed to exist in a (possibly very short) time interval around $t = 0$.

For example, consider the case where $x(0) = 0$. Then the problem can be solved analytically by separation of variables:

$$\int \frac{dx}{1 + x^2} = \int dt,$$

which yields

$$\tan^{-1} x = t + C$$

The initial condition $x(0) = 0$ implies $C = 0$. Hence $x(t) = \tan t$ is the solution. But notice that this solution exists only for $-\pi/2 < t < \pi/2$, because $x(t) \to \pm\infty$ as $t \to \pm\pi/2$. Outside of that time interval, there is no solution to the initial value problem for $x_0 = 0$. ■

The amazing thing about Example 2.5.2 is that the system has solutions that reach infinity *in finite time*. This phenomenon is called ***blow-up***. As the name suggests, it is of physical relevance in models of combustion and other runaway processes.

There are various ways to extend the existence and uniqueness theorem. One can allow f to depend on time t, or on several variables $x_1, \ldots, x_n$. One of the most useful generalizations will be discussed later in Section 6.2.

From now on, we will not worry about issues of existence and uniqueness—our vector fields will typically be smooth enough to avoid trouble. If we happen to come across a more dangerous example, we'll deal with it then.

2.6 Impossibility of Oscillations

Fixed points dominate the dynamics of first-order systems. In all our examples so far, all trajectories either approached a fixed point, or diverged to $\pm\infty$. In fact, those are the *only* things that can happen for a vector field on the real line. The reason is that trajectories are forced to increase or decrease monotonically, or remain constant (Figure 2.6.1). To put it more geometrically, the phase point never reverses direction.

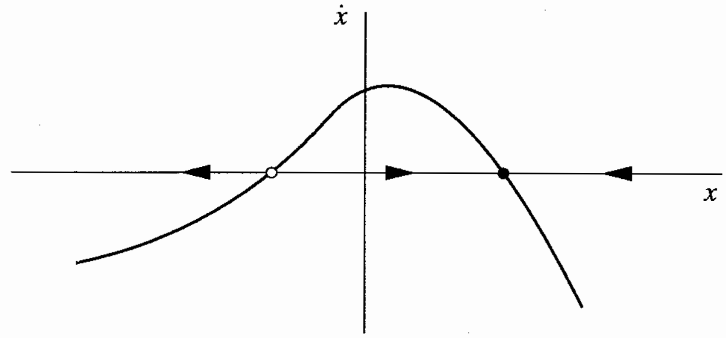

Figure 2.6.1

Thus, if a fixed point is regarded as an equilibrium solution, the approach to equilibrium is always *monotonic*—overshoot and damped oscillations can never occur in a first-order system. For the same reason, undamped oscillations are impossible. Hence *there are no periodic solutions to* $\dot{x} = f(x)$.

These general results are fundamentally topological in origin. They reflect the fact that $\dot{x} = f(x)$ corresponds to flow on a *line.* If you flow monotonically on a line, you'll never come back to your starting place—that's why periodic solutions are impossible. (Of course, if we were dealing with a *circle* rather than a line, we *could* eventually return to our starting place. Thus vector fields on the circle can exhibit periodic solutions, as we discuss in Chapter 4.)

Mechanical Analog: Overdamped Systems

It may seem surprising that solutions to $\dot{x} = f(x)$ can't oscillate. But this result becomes obvious if we think in terms of a mechanical analog. We regard $\dot{x} = f(x)$ as a limiting case of Newton's law, in the limit where the "inertia term" $m\ddot{x}$ is negligible.

For example, suppose a mass m is attached to a nonlinear spring whose restoring force is $F(x)$, where x is the displacement from the origin. Furthermore, suppose that the mass is immersed in a vat of very viscous fluid, like honey or motor oil (Figure 2.6.2), so that it is subject to a damping force $b\dot{x}$. Then Newton's law is

$$m\ddot{x} + b\dot{x} = F(x).$$

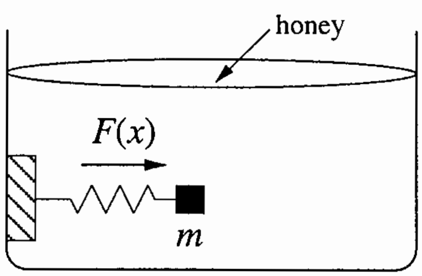

Figure 2.6.2

If the viscous damping is strong compared to the inertia term ($b\dot{x} >> m\ddot{x}$), the system should behave like $b\dot{x} = F(x)$, or equivalently $\dot{x} = f(x)$, where $f(x) = b^{-1}F(x)$. In this ***overdamped*** limit, the behavior of the mechanical system is clear. The mass prefers to sit at a stable equilibrium, where $f(x) = 0$ and $f'(x) < 0$. If displaced a bit, the mass is slowly dragged back to equilibrium by the restoring force. No overshoot can occur, because the damping is enormous. And undamped oscillations are out of the question! These conclusions agree with those obtained earlier by geometric reasoning.

Actually, we should confess that this argument contains a slight swindle. The neglect of the inertia term $m\ddot{x}$ is valid, but only after a rapid initial transient during which the inertia and damping terms are of comparable size. An honest discussion of this point requires more machinery than we have available. We'll return to this matter in Section 3.5.

2.7 Potentials

There's another way to visualize the dynamics of the first-order system $\dot{x} = f(x)$, based on the physical idea of potential energy. We picture a particle sliding down the walls of a potential well, where the ***potential*** $V(x)$ is defined by

$$f(x) = -\frac{dV}{dx}.$$

As before, you should imagine that the particle is heavily damped—its inertia is completely negligible compared to the damping force and the force due to the potential. For example, suppose that the particle has to slog through a thick layer of goo that covers the walls of the potential (Figure 2.7.1).

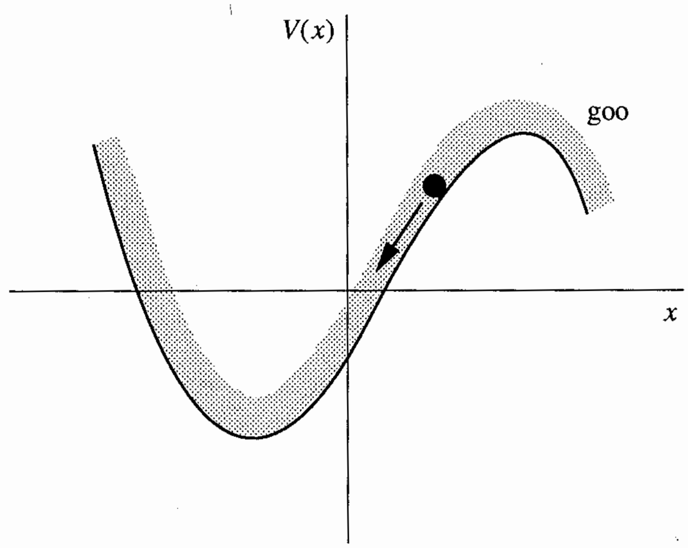

Figure 2.7.1

The negative sign in the definition of V follows the standard convention in physics; it implies that the particle always moves "downhill" as the motion proceeds. To see this, we think of x as a function of t, and then calculate the time-derivative of $V(x(t))$. Using the chain rule, we obtain

$$\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}.$$

Now for a first-order system,

$$\frac{dx}{dt} = -\frac{dV}{dx},$$

since $\dot{x} = f(x) = -dV/dx$, by the definition of the potential. Hence,

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2 \leq 0.$$

Thus $V(t)$ *decreases along trajectories,* and so the particle always moves toward lower potential. Of course, if the particle happens to be at an ***equilibrium*** point where $dV/dx = 0$, then V remains constant. This is to be expected, since $dV/dx = 0$ implies $\dot{x} = 0$; equilibria occur at the fixed points of the vector field. Note that local minima of $V(x)$ correspond to *stable* fixed points, as we'd expect intuitively, and local maxima correspond to unstable fixed points.

EXAMPLE 2.7.1:

Graph the potential for the system $\dot{x} = -x$, and identify all the equilibrium points.

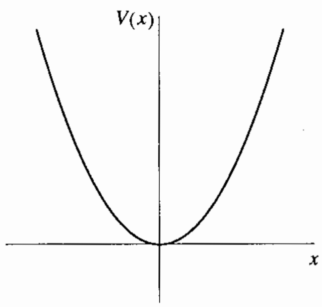

Figure 2.7.2

Solution: We need to find $V(x)$ such that $-dV/dx = -x$. The general solution is $V(x) = \frac{1}{2}x^2 + C$, where C is an arbitrary constant. (It always happens that the potential is only defined up to an additive constant. For convenience, we usually choose $C = 0$.) The graph of $V(x)$ is shown in Figure 2.7.2. The only equilibrium point occurs at $x = 0$, and it's stable. ■

EXAMPLE 2.7.2:

Graph the potential for the system $\dot{x} = x - x^3$, and identify all equilibrium points.

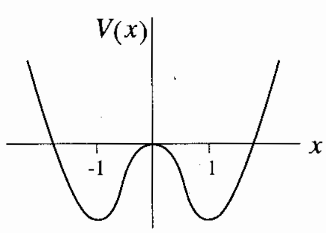

Figure 2.7.3

Solution: Solving $-dV/dx = x - x^3$ yields $V = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + C$. Once again we set $C = 0$. Figure 2.7.3 shows the graph of V. The local minima at $x = \pm 1$ correspond to stable equilibria, and the local maximum at $x = 0$ corresponds to an unstable equilibrium. The potential shown in Figure 2.7.3 is often called a ***double-well potential***, and the system is said to be ***bistable***, since it has two stable equilibria. ■