Three Bearing Method for Passive Triangulation in Systems With Unknown Deterministic Biases

Abstract

An operationally simple method for triangulation in systems with strong biases is presented. The method requires three bearing measurements and two turns by the aircraft. Analysis of the method shows that accurate triangulation can be performed without knowing the bias function. The method has been tested by numerical simulation and in actual flights, and the test results are reported. An approximate method for including small random fluctuations and confidence contours about the point of triangulation is given.

I. Introduction

Many airborne and surface sensing systems in present use are passive devices. As the search platform moves through space, a series of passive bearings is taken and the position of the source (henceforth called the target) is estimated by triangulation. The observed bearing θ_0 is generally composed of three terms

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$$\theta_0 = \theta_r + b(\theta_r, \mathbf{f}) + \epsilon w \ . \qquad (1)$$

In this equation θ_r is the real bearing between the searcher's heading and the target (measured clockwise), $b(\theta_r, \mathbf{f})$ is a *deterministic* bias function depending upon θ_r and the system parameters f, and ϵw is a random noise term, with $E(w) = 0$ and $E(w^2) = \sigma^2 = 0(1)$, so that ϵ is a measure of the intensity of the random terms. In many airborne platforms, $|b(\theta_r, \mathbf{f})| \sim 10° - 30°$, but $\epsilon\sigma \sim 2° - 4°$, so that the deterministic biases are much stronger than the random terms.

In this correspondence, we consider a triangulation problem in the search for a stationary target (or almost stationary, relative to the searcher) in which deterministic and random terms enter into θ_0. We assume that the sensing platform is an aircraft. The treatment of random errors in triangulation received considerable attention over the last 20 years (some examples are [1]-[5]) and various linear and nonlinear filters were developed for passive localization problems. In general, these papers treat the *statistical* problem connected with (1). In that case, the bias is ignored or presumed known, many bearings are taken and some kind of statistical procedure is sought to obtain a best estimate of position. In this correspondence, we study the case of a strong, but unknown deterministic bias. In order to demonstrate that deterministic biases may be important, consider the simplest triangulation problem, shown in Fig. 1. At time 0, the first bearing, θ_{01} is observed. The aircraft flies a distance S and then a second bearing θ_{02} is observed. It is easy to see that the distance to the target after the second bearing measurement R satisfies

$$R/S = \sin(\theta_{01})/\sin(\theta_{02} - \theta_{01}) \ . \qquad (2)$$

Assume that the observed bearing is the sum of the real bearing θ_r and a deterministic bias function $b(\theta_r)$. Then in (2),

$$\theta_{0i} = \theta_{ri} + b(\theta_{ri}) \ . \qquad (3)$$

Fig. 2 shows the relative error obtained using (2) with $b(\theta_r) = 10\sin(3\theta_r)$ as a function of θ_{02}, for $\theta_{01} = 15°$ and $35°$. The relative error can be as large as 50 percent. In real systems $b(\theta_r, \mathbf{f})$ is not a simple sinusoid; in fact, it may not be known at all.

In this correspondence, a method for passive triangulation in systems with a bias is introduced. The method has these features:

1) Operationally, it is simple to use (and has been, see Section IV).
2) The bias function need not be known at all.
3) The method is stable to small perturbations in measurements.
4) Random errors can be incorporated into the method.

In Section II, the operational method is described. The main analytical results, for $\epsilon = 0$, are given in Section

III and numerical and flight testing is reported in Section IV. In Section V, it is shown how the method can be modified to include random fluctuations, when ϵ is small.

II. Three Bearing Method: Operational Description

The geometry of the three bearing method is shown in Fig. 3. The method consists of the following operations:

1) An initial value of the observed bearing $\theta_{01} = \alpha$ is recorded.
2) The aircraft turns through an angle Δ_1. If the observed bearing is positive (negative), then the turn is positive (negative). The aircraft flies until the observed bearing θ_{02} is equal to α. The distance traveled, s_1, is recorded;
3) The aircraft turns again, through an angle Δ_2, and flies until the observed bearing θ_{03} is equal to α. The distance traveled, s_2, is recorded,

In Section III, we show that from Δ_1, Δ_2, s_1', and s_2 the real range and bearing to the target can be calculated, without knowing the bias function. The turn angles Δ_1 and Δ_2 are free parameters that can be determined by the operator. A large turn angle implies a long time of flight until the original bearing is observed again and more accurate calculations. Conversely, a small turn implies a short leg, but the results will be less accurate.

The three bearing method has a number of operational virtues. The required measurements are Δ_1, Δ_2, s_1, and s_2. The turn angles can be measured accurately with the aircraft's compass. The distances s_1 and s_2 can be measured by clocking travel times and multiplying by the aircraft speed. Thus all measurements can be made very precisely.

III. Three Bearing Method: Analytical Results

In this section we derive the main results of the paper. Assume that Δ_1, Δ_2, s_1, and s_2 are known. Set $\gamma = s_1/s_2$ and $\eta = \sin \Delta_2 / \sin \Delta_1$. Let R_1 be the distance to the target when second bearing θ_{02} is observed (see Fig. 4), R_2 be the distance to the target when θ_{03} is observed, and θ_r be the real bearing to the target when θ_{03} is observed. Then we obtain the following.

Result 1: Assume that γ and η are known. In the coordinate system shown in Fig. 4, the following results hold.

If $\theta_0(\theta_r)$ is a single valued function, then

$$R_1 = s_1/[1 + \eta^2\gamma^2 - 2 \cos \eta\Delta_1]^{1/2} \tag{4}$$

$$R_2 = s_2 \sin(\theta_r - \Delta_2)/\sin \Delta_2 \tag{5}$$

$$\sin \theta_r = (R_1/s_2) \sin \Delta_2 \tag{6}$$

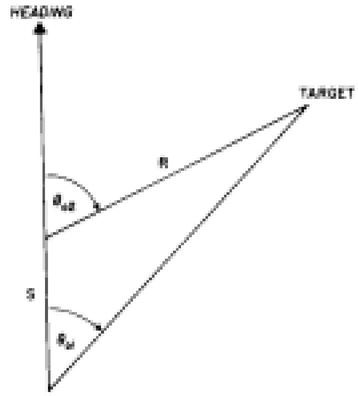


Fig. 1. Setup for simplest triangulation problem.

Fig. 2. Percentage error in range estimation for simplest triangulation problem with 10° bias function. (A) $\theta_{01} = 35°$. (B) $\theta_{01} = 15°$. Percent error $((R_{obs} - R_{real})/R_{obs}) \times 100$.

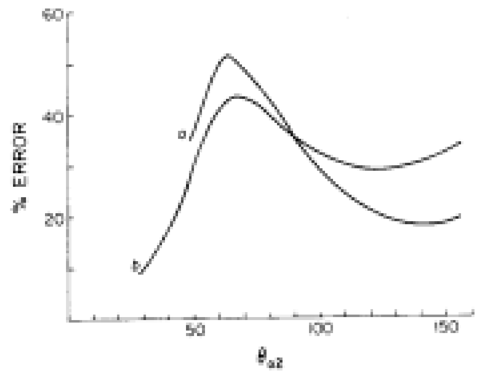
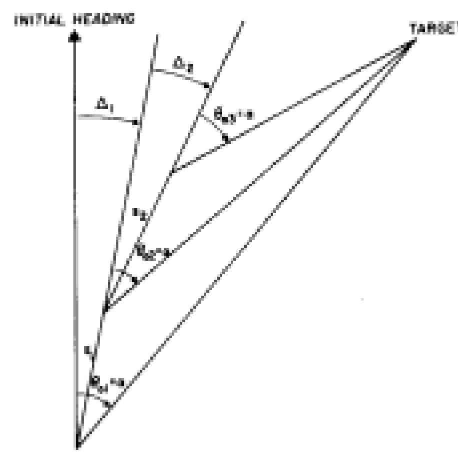


Fig. 3. Operational illustration of three bearing method.



$$\tan \theta_r = [-s_2 \tan \Delta_2 \tan(\Delta_1 + \Delta_2) - (s_1 \sin \Delta_2 - (s_2 + s_1 \cos \Delta_2) \tan(\Delta_1 + \Delta_2) \tan \Delta_2)] / [-s_2 \tan(\Delta_1 + \Delta_2) + \tan \Delta_2 (s_2 + s_1 \cos \Delta_2 + s_1 \sin \Delta_2 \tan(\Delta_1 + \Delta_2))] . \tag{7}$$

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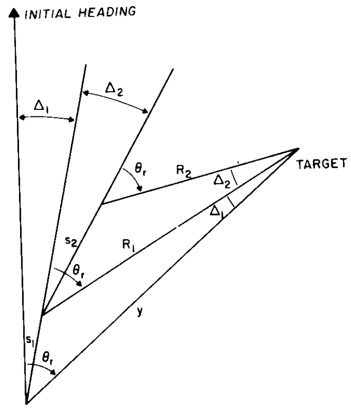
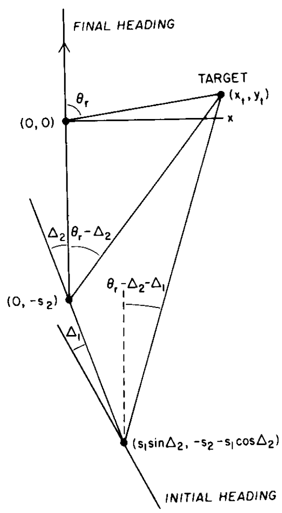


Fig. 4. Analytical setup for solution of three bearing method.

Fig. 5. Analytical setup for solution of three bearing method in Cartesian coordinates.



Note that (6) by itself cannot be used to determine θ_r, since the arcsin is double valued. However, (6) and (7) taken together allow one to determine θ_r uniquely.

To prove the above result, we use Fig. 4. That the interior angles on the two triangles are Δ_1 and Δ_2 can be shown by adding angles and insuring that the sum is π.

Referring to Fig. 4 and using the law of sines twice shows that

$$s_1/\sin\Delta_1 = y/\sin\theta_r = R_1/\sin(\theta_r - \Delta_1)$$
$$s_2/\sin\Delta_2 = R_1/\sin\theta_r = R_2/\sin(\theta_r - \Delta_2) \quad . \tag{8}$$

For the triangle with sides s_1, y, and R_1,

$$\cos\Delta_1 = (y^2 + R_1{}^2 - s_1{}^2)/2yR_1 \quad . \tag{9}$$

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From (8),

$$y = s_1 \sin\theta_r/\sin\Delta_1, \quad \sin\theta_r = R_1 \sin\Delta_2/s_2 \tag{10}$$

so that

$$y = (s_1/s_2) R_1 (\sin\Delta_2/\sin\Delta_1) = \gamma\eta R_1 \quad . \tag{11}$$

Using (11) in (9) gives

$$\cos\Delta_1 = (\gamma^2\eta^2 R_1{}^2 + R_1{}^2 - s_1{}^2)/2\gamma\eta R_1{}^2 \quad . \tag{12}$$

Equation (12) is cast into a dimensionless form by setting $r_1 = R_1/s_2$. Then after some simplification, (12) becomes

$$r_1 = \gamma/[\gamma^2\eta^2 + 1 - 2\gamma\eta\cos\Delta_1]^{1/2} \quad . \tag{13}$$

If one returns to physical variables, (13) implies (4).

From (8), $\sin\theta_r = r_1 \sin\Delta_2$ so that (6) is verified.

To find R_2, equivalently $r_2 = R_2/s_2$, we again use (8) to obtain

$$R_2 = R_1 \sin(\theta_r - \Delta_2)/\sin\theta_r \quad . \tag{14}$$

Since $\sin\theta_r = R_1 \sin\Delta_2/s_2$, (14) is equivalent to

$$R_2 = s_2 \sin(\theta_r - \Delta_2)/\sin\Delta_2 \tag{15}$$

and this verifies (5).

In order to verify (7), introduce a Cartesian coordinate system centered at the aircraft after the third bearing measurement with the y-axis along the aircraft heading. Let (x_t, y_t) be the position of the target in this coordinate system (see Fig. 5):

$$x_t = y_t \tan\theta_r \tag{16a}$$
$$x_t = (y_t + s_2)\tan(\theta_r - \Delta_2) \tag{16b}$$
$$x_t - s_1 \sin\Delta_2 = (y_t + s_2 + s_1 \cos\Delta_2)\tan(\theta_r - \Delta_1 - \Delta_2) \quad . \tag{16c}$$

Using the trigonometric addition formula and (16a), equations (16b) and (16c) can be rewritten as

$$x_t[1 + (x_t/y_t)\tan\Delta_2] = (y_t + s_2)(x_t/y_t - \tan\Delta_2) \tag{17a}$$
$$(x_t - s_1 \sin\Delta_2)[1 + (x_t/y_t)\tan(\Delta_1 + \Delta_2)]$$
$$= (y_t + s_2 + s_1\cos\Delta_2)[x_t/y_t - \tan(\Delta_1 + \Delta_2)] \quad . \tag{17b}$$

When (17a) is multiplied by $\tan(\Delta_1 + \Delta_2)$ and (17b) is multiplied by $\tan\Delta_2$ and (17b) is subtracted from (17a), we obtain

$$x_t\{-s_2\tan(\Delta_1+\Delta_2)+[s_2+s_1\cos\Delta_2 + s_1\sin\Delta_2\tan(\Delta_1+\Delta_2)]\tan\Delta_2\}$$
$$=y_t\{-s_2\tan\Delta_2\tan(\Delta_1+\Delta_2)-[s_1\sin\Delta_2 - (s_2+s_1\cos\Delta_2)\tan(\Delta_1+\Delta_2)]\tan\Delta_2\} \ . \qquad (18)$$

In light of (16a), equation (18) verifies (7).

Note that (6) and (7) can be combined to give the result

$$\cot\theta_r=\cot\Delta_1-(s_2/s_1)\csc\Delta_2 \ . \qquad (19)$$

Result 2: The three bearing method is stable. Small changes in s_1, s_2, Δ_1, and/or Δ_2 produce small changes in r_1, r_2, and θ_r.

In order to demonstrate this result, we show that the derivatives of r_1, θ_r, and r_2 with respect to γ and η are bounded.

From (13) one finds

$$\partial r_1/\partial\gamma=(1-\gamma\eta\cos\Delta_1)/[\gamma^2\eta^2+1-2\gamma\eta\cos\Delta_1]^{3/2}$$
$$\partial r_1/\partial\eta=-\gamma^2(\gamma\eta-\cos\Delta_1)/[\gamma^2\eta^2+1-2\gamma\eta\cos\Delta_1]^{3/2} \qquad (20)$$

Using (6a) gives after some simplification

$$\partial\theta_r/\partial\gamma=(1-r_1{}^2\sin\Delta_2{}^2)^{-1/2}(\partial r_1/\partial\gamma)$$
$$\partial\theta_r/\partial\eta=(1-r_1{}^2\sin\Delta_2{}^2)^{-1/2}(\partial r_1/\partial\eta) \ . \qquad (21)$$

Inspection of (20) and (21) shows that all the derivatives are well behaved functions of γ and η as long as the turn angles Δ_1 and Δ_2 are bounded away from zero.

IV. Tests of the Three Bearing Method

In this section we report on operational flight tests and numerical simulations used to evaluate the three bearing method.

In July, 1980, the three bearing method was operationally tested in flights by Tactical Electronic Warfare Squadron 136 (VAQ-136) of the U.S. Naval Air Force. The use of the three bearing method in this operational test gave results which were accurate to within less than 10 percent error in range. In one case, the range error was less than 3 percent. Further tests are now in progress and those results will be reported elsewhere.

A simulation was also used to test the three bearing method. The bias function used was a constant 10° and the searcher started at the origin. The simulation was written so that bearing measurements were taken to be accurate to within ± 0.5°. Table 1 shows the results of some of these simulations.

If, instead of the three bearing method, a simple triangulation were performed, then after the first turn and the second bearing measurement there is enough information available for triangulation and calculation of R_1. Table II shows the real value of R_1 and the values calculated by the three bearing method and by triangulation.

V. Incorporating Small Random Errors in the Three Bearing Method

Real bearing measurements always involve random errors, in addition to deterministic biases. This section shows how some of the random effects can be treated when the intensity of the random noise is small. When noise is present, the real bearing is no longer θ_r, but is $\theta_r+\epsilon w$. If $\theta_0+\epsilon u$ denotes the observed bearing, then

$$\theta_0+\epsilon u=\theta_r+\epsilon w+b(\theta_r+\epsilon w) \ . \qquad (22)$$

By Taylor expanding (22), we see that $u\cong[1+b'(\theta_r)]w$ so that the observed equipment error, var(u), is related to the noise by var(u) $\cong[1+b'(\theta_r)]^2$ var(w).

In order to incorporate the noise, in (8) replace θ_r by $\theta_r+\epsilon w_i$, with the appropriate value of i determined by referring to Fig. 4. This gives

$$s_1/\sin[\Delta_1+\epsilon(w_2-w_1)]=y/\sin(\theta_r+\epsilon w_2) =R_1/\sin(\theta_r+\epsilon w_1-\Delta_1)$$
$$s_2/\sin[\Delta_2+\epsilon(w_3-w_2)]=R_1/\sin(\theta_r+\epsilon w_3) =R_2/\sin(\theta_r+\epsilon w_2-\Delta_2) \ . \qquad (8')$$

Note that all three noise terms appear in (8′), as one would expect.

Proceeding as in Section III, we obtain equations for θ_r, R_1, and R_2. These are

$$\theta_r=\arcsin\{(R_1/s_2)\sin[\Delta_2+\epsilon(w_3-w_2)]\}-\epsilon w_3 \qquad (23)$$

$$\cos\Delta_1=(s_1{}^2\sin^2(\theta_r+\epsilon w_2) + (R_1{}^2-s_1{}^2)\{\sin^2[\Delta_1+\epsilon(w_2-w_1)]\})/ \{2R_1s_1\sin(\theta_r+\epsilon w_2)\sin[\Delta_1+\epsilon(w_2-w_1)]\} \qquad (24)$$

$$R_2=s_2\sin(\theta_r+\epsilon w_2-\Delta_2)/\sin[\Delta_2+\epsilon(w_3-w_2)] \ . \qquad (25)$$

If $\epsilon=0$, then (23)-(25) reduce to (10), (12), and (15) respectively. Let $\bar\theta_r$, $\bar R_1$, and $\bar R_2$ be the solutions of

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TABLE I
Numerical Test of the Three Bearing Method

Emitter Location	α (deg)	$\Delta_1=\Delta_2$ (deg)	s_1	s_2	(R_2, θ_2) By Three Bearing Method	Real R_2, θ_2
(−100, 100)	−35	−15	51.66	36.7	(71.4, 134.3)	(70.7, 135)
(100, 100)	55	15	51.66	36.7	(71.4, 45.3)	(70.7, 41)
(100, 200)	37	15	128	59.4	(48.7, 27)	(44.5, 26)
(200, 100)	73	15	65	24	(153, 62)	(157, 63)
(−100, −100)	235	−15	52	63	(211, 225)	(212, 225)

TABLE II
Values of R_1 by Triangulation and the Three Bearing Method

Emitter Location	Real R_1	R_1 by Three Bearing Method	Percent Error	R_1 by Triangulation	Percent Error
(−100, 100)	100.1	100.7	0.6	68.3	46.7
(100, 100)	100.1	100.7	0.6	68.3	46.6
(100, 200)	101.2	105.1	4.8	182	44.4
(200, 100)	187	184	1.6	213	12.2
(−100, −100)	173	173	0	187	7.5

(10), (12), (15) and set $\theta_r = \bar{\theta}_r + \epsilon\beta$, $R_1 = \bar{R}_1 + \epsilon r_1$, and $R_2 = \bar{R}_2 + \epsilon r_2$. Differentiating (23)-(25) with respect to ϵ and setting $\epsilon = 0$ gives

$$r_2 = (s_2 \cos\bar{\theta}_r/\sin\Delta_2)(w_2+\beta) - \bar{R}_2(\cot\Delta_2)(w_3 - w_2) \tag{26}$$

$$\beta = (\tan\Delta_2/s_2)\,r_1 + (\bar{R}_1/s_2)(w_3 - w_2) - w_3 \tag{27}$$

$$r_1 = (A_1 - A_2)/A_3 \tag{28}$$

where

$$A_1 = [s_1^2\sin^2\bar{\theta}_r + (\bar{R}_1^2 - \bar{s}_1^2)\sin^2\Delta_1] \cdot \{2\bar{R}_1 s_1\cos\bar{\theta}_r\sin\Delta_1\,[\bar{R}/s_2(w_3 - w_2) + w_2 - w_3] + 2\bar{R}_1 s_1\sin\bar{\theta}_r(\cos\Delta_1)(w_2 - w_1)\} \tag{29}$$

$$A_2 = (2\bar{R}_1 s_1\sin\bar{\theta}_r\sin\Delta_1)\,[2s_1^2\sin\bar{\theta}_r\cos\bar{\theta}_r w_2 + 2(\bar{R}_1^2 - \bar{s}_1^2)\sin\Delta_1(\cos\Delta_1)(w_2 - w_1)] \tag{30}$$

$$A_3 = (2\bar{R}_1 s_1\sin\bar{\theta}_r\sin\Delta_1) \cdot [2(\tan\Delta_2/s_2)s_1^2\sin\bar{\theta}_r\cos\bar{\theta}_r + 2\bar{R}_1\sin^2\Delta_1] - [s_1^2\sin^2\bar{\theta}_r + (\bar{R}_1^2 - \bar{s}_1^2)\sin^2\Delta_1] \cdot [2s_1\sin\bar{\theta}_r\sin\Delta_1 - 2(\tan\Delta_2/s_2)\bar{R}_1 s_1\cos\bar{\theta}_r\sin\Delta_1]\,. \tag{31}$$

Once the statistics of the w_i are known, the statistics of r_1, r_2, and β can be found in a straightforward manner.

Acknowledgment

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MARC MANGEL
Department of Mathematics
University of California
Davis, CA 95616